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Strength of convergence in non-free transformation groups ☆

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Abstract

Let (G, X) be a transformation group where the group G does not necessarily act freely on the space X. We investigate the extent to which the action of G may fail to be proper. Stability subgroups are used to define new notions of strength of convergence in the orbit space and of measure accumulation along orbits. By using the representation theory of the associated crossed product C^*-algebra, we show that these notions are equivalent under certain conditions.

Keywords: Transformation group; Stability subgroup; Orbit space; Proper action; k-Times convergence; Measure accumulation; Crossed-product C^*-algebra; Induced representation; Spectrum of a C^*-algebra; Multiplicity of a representation

1. Introduction

Let (G, X) be a second-countable, locally compact, Hausdorff transformation group, so that the group G acts continuously on the space X. Thinking of the group action as time evolution on a state space, an action is proper if states move far away from their original position over long periods of time. We showed in [3] that the failure of properness in a free action of G on X can

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be counted by two methods, one topological and the other measure-theoretic, and that they give the same answer. The topological method is based on the notion of strength of convergence from [2, Definition 2.2] and is motivated by the behaviour of sequences in the dual of a nilpotent Lie group [17] and by several previous considerations of the failure of properness [14,16,18,23]. For example, a sequence $(x_n)_n$ in X *converges* 2*-times in the orbit space to* z *in* X if there are two sequences $(t_n^{(1)})_n$ and $(t_n^{(2)})_n$ in G such that $(t_n^{(1)} \cdot x_n)_n$ and $(t_n^{(2)} \cdot x_n)_n$ both converge to z, and $t_n^{(2)}(t_n^{(1)})^{-1} \to \infty$. The measure-theoretic method involves the accumulation of Haar measure as in [16]. These methods are linked via the representation theory of the crossed product C^*-algebra $C_0(X) \rtimes G$, and in particular via the lower relative multiplicity number $M_L(\pi, (\pi_n))$ associated to a particular convergent sequence $\pi_n \to \pi$ in the spectrum of $C_0(X) \rtimes G$.

There are two natural ways to generalise these results. First, we can replace X by a 'non-commutative space', that is, replace $C_0(X)$ by a non-commutative C^*-algebra A on which G acts by automorphisms; we investigated this in [4] under the assumption that the induced action of G on the primitive ideal space of A is free. Second, we can retain the transformation group (G, X) and relax the assumption that the action of G on X should be free. In this paper we focus on non-free actions of G on X.

When the action is not free, the technical difficulties of working with induced representations of the crossed product $C_0(X) \rtimes G$ increase substantially [8–10,15,20,22,23]. For this reason, and because we want to make use of the dual action on $C_0(X) \rtimes G$, we assume that the group G is abelian in Sections 4–6.

In Section 2 we discuss our set-up which involves careful choices of measures on the subgroups and quotients of G. In Section 3 we define k-times convergence in the presence of stability subgroups and show that measure accumulation gives rise to sequences which converge k-times. In Section 4 we introduce induced representations of the crossed product $C_0(X) \rtimes G$. We consider a sequence of induced representations $(\pi_n)_n$ converging to an induced representation π, and establish sufficient conditions involving k-times convergence which ensure that $M_L(\pi, (\pi_n))$ is bounded below. In Section 5 we establish upper bounds on $M_L(\pi, (\pi_n))$ arising from bounds on measure accumulation. In Section 6 we combine our results to obtain our main theorem (Theorem 6.1) which shows that, under certain conditions, strength of convergence and measure accumulation in the transformation group (G, X) are equivalent, being linked by the representation theory of $C_0(X) \rtimes G$.

Preliminaries. Let A be a C^*-algebra, $\hat{A}$ its spectrum and $\pi \in \hat{A}$ an irreducible representation. Upper and lower multiplicities $M_U(\pi)$ and $M_L(\pi)$, and upper and lower multiplicities $M_U(\pi, (\pi_n))$ and $M_L(\pi, (\pi_n))$ relative to a net (π_n) in $\hat{A}$ were first defined in [1] and [5], respectively. We refer the reader to [3, §2] for a convenient summary of what is needed here. We set $\mathbb{P} = \mathbb{N} \setminus \{0\}$.

2. The set-up: Choices of measures on the subgroups of G

Let G be a locally compact group with left Haar measure μ. Let Σ be the family of all closed subgroups of G. We endow Σ with the *Fell topology* from [11]. A basis for this topology is the family of sets

$$U(C, F) = \{H \in \Sigma\colon\ C \cap H = \emptyset \text{ and } H \cap A \neq \emptyset \text{ for each } A \in F\},$$

where C is a compact subset of G and F a finite family of non-empty open subsets of G. Then Σ is a compact Hausdorff space by [11, Theorem 1 and Remark IV]. We will frequently use that $H_\lambda \to H$ in Σ if and only if

(1) if $h \in H$ then there exist a subnet $(H_{\lambda(\mu)})$ and $h_\mu \in H_{\lambda(\mu)}$ such that $h_\mu \to h$, and
(2) if $h_\lambda \in H_\lambda$ and $h_\lambda \to h$ then $h \in H$

(see, for example, [24, Lemma H2]).

Fix a function $f_0 \in C_c(G)$ with $f_0(e) = 1$ and $0 \leqslant f_0 \leqslant 1$. For each $H \in \Sigma$ we choose the left Haar measure α_H on H satisfying

$$\int_H f_0(t)\, d\alpha_H(t) = 1.$$

Such a choice of measures is called a *continuous choice of Haar measures* on the closed subgroups of G, and has the property that

$$H \mapsto \int_H f(t)\, d\alpha_H(t)$$

is a continuous function on Σ for any $f \in C_c(G)$ (see [13, p. 908] or [24, Lemma H.8]). We write Δ_H for the modular function associated with the measure α_H.

Now let (G, X) be a transformation group. The *stability subgroup* at $x \in X$ is $S_x := \{s \in G: s \cdot x = x\}$. We write α_x for α_{S_x}, $q_x : G \to G/S_x$ for the quotient map, and $\dot{s} = sS_x = q_x(s)$ for $s \in G$. We also define $\phi_x : G \to X$ by $\phi_x(s) = s \cdot x$.

If H is a normal subgroup of G then there exists a unique right-invariant Haar measure ν_H on G/H such that for all $f \in C_c(G)$,

$$\int_G f(s)\, d\mu(s) = \int_{G/H} \int_H f(st)\, d\alpha_H(t)\, d\nu_H(\dot{s}) \tag{2.1}$$

(see, for example, [21, Appendix C]). If $H = S_x$ then we write ν_x for ν_{S_x}. We claim that it follows that if χ_E is the characteristic function of a measurable subset E of G, then

$$\int_G \chi_E(s) f(s)\, d\mu(s) = \int_{G/H} \int_H \chi_E(st) f(st)\, d\alpha_H(t)\, d\nu_H(\dot{s}). \tag{2.2}$$

Indeed, since $\operatorname{supp} f$ is compact, we may assume that E has finite measure so that by Lusin's Theorem χ_E is the pointwise limit (almost everywhere) of a sequence of continuous functions $g_n : G \to [0, 1]$. Then the claim follows by repeated applications of the Dominated Convergence Theorem.

Typically we will be interested in sequences $(x_n)_n$ in X such that $S_{x_n} \to S_z$ for some $z \in X$ as $n \to \infty$, and that S_{x_n} and S_z are normal in G; we then assume that the quotient measures ν_{x_n} and ν_z have been chosen to satisfy (2.1). When $x_n \to z$ in X, the assumption that $S_{x_n} \to S_z$ is, of course, weaker than the assumption that the stability subgroups vary continuously over the whole space X.

3. *k*-Times convergence with stability

The following definition of k-times convergence immediately reduces to the one used in [2,3] when the action of G on X is free.

Definition 3.1. Let (G, X) be a transformation group. A sequence $(x_n)_{n\geqslant 1}$ in X is said to *converge k-times in X/G to $z \in X$* if there exist k sequences $(t_n^{(1)})_n, (t_n^{(2)})_n, \dots, (t_n^{(k)})_n$ in G such that

(1) $t_n^{(i)} \cdot x_n \to z$ as $n \to \infty$ for $1 \leqslant i \leqslant k$, and
(2) if $1 \leqslant i < j \leqslant k$ then $t_n^{(j)}(t_n^{(i)})^{-1} S_{x_n} \to \infty$ as $n \to \infty$ (that is, for every compact subset K of G, $t_n^{(j)}(t_n^{(i)})^{-1} S_{x_n}$ is eventually disjoint from K).

As in the free case, k-times converging sequences in X/G arise from measure accumulation. We show this in Proposition 3.5 below, and the next three lemmas lead towards this. In order to describe measure accumulation for a point $z \in X$, we require that z has a base of neighbourhoods V such that $q_z(\phi_z^{-1}(V))$ has finite Haar measure in G/S_z, and by Lemma 3.2 this is equivalent to the orbit $G \cdot z$ being locally closed in X. Lemmas 3.3 and 3.4 are technical ones addressing the following issue: if W is a compact subset of G and $S_{x_n} \to S_z$ in Σ, then we want to compare the measures $\nu_{x_n}(q_{x_n}(W S_{x_n}))$ and $\nu_z(q_z(W S_z))$ for large n. But even though $W S_{x_n} \to W S_z$ in the Fell topology on the closed subsets of G, it is conceivable that $\chi_{W S_{x_n}}$ may not converge pointwise almost everywhere to $\chi_{W S_z}$.

Lemma 3.2. *Let (G, X) be a second-countable transformation group. Let $z \in X$ and suppose that the stability subgroup S_z is normal in G. Then the following are equivalent*:

(1) *the orbit $G \cdot z$ is not locally closed in X*;
(2) *for every $k \in \mathbb{P}$, the sequence $z, z, z, \dots$ converges k-times in X/G to z*;
(3) *for every open neighbourhood V of z, $\nu_z(q_z(\phi_z^{-1}(V))) = \infty$*;
(4) *for every open neighbourhood V of z, $q_z(\phi_z^{-1}(V))$ is not relatively compact in G/S_z.*

Proof. Let $(V_n)_{n\geqslant 1}$ be a decreasing basic sequence of open neighbourhoods of z in X and let $(K_n)_{n\geqslant 1}$ be an increasing sequence of compact subsets of G such that $G = \bigcup_{n\geqslant 1} \mathrm{Int}(K_n)$.

(1) $\Rightarrow$ (2). Suppose that $G \cdot z$ is not locally closed. Then $W \cap (\overline{G \cdot z} \setminus G \cdot z) \neq \emptyset$ for every neighbourhood W of z. Let $k \geqslant 1$. We will construct the required k sequences $(t_n^{(i)})_{n\geqslant 1}$ in G by induction.

Let $n \in \mathbb{P}$. We construct $t_n^{(i)}$ as follows. Let $t_n^{(1)} = e$. Since $G \cdot z$ is not locally closed there exists $y \in V_n \cap (\overline{G \cdot z} \setminus G \cdot z)$. Since y is in the closure of $G \cdot z$ and V_n is open, a straightforward compactness argument shows that given any compact subset K of G there exists $t_K \in G \setminus K S_z$ such that $t_K \cdot z \in V_n$. So there exists $t_n^{(2)} \in G \setminus K_n S_z = G \setminus K_n t_n^{(1)} S_z$ such that $t_n^{(2)} \cdot z \in V_n$. Proceeding inductively we obtain $t_n^{(2)}, t_n^{(3)} \dots, t_n^{(k)}$ such that

$$t_n^{(j)} \cdot z \in V_n \quad \text{and} \quad t_n^{(j)} \in G \setminus \left(\bigcup_{i=1}^{j-1} K_n t_n^{(i)} S_z \right)$$

for $2 \leqslant j \leqslant k$.

Since $(V_n)_{n\geqslant 1}$ is a decreasing basic sequence of open neighbourhoods of z, $t_n^{(j)} \cdot z \to z$ as $n \to \infty$ for $1 \leqslant j \leqslant k$. By way of contradiction, suppose that for some $i < j$ there exists a compact subset K of G such that $t_n^{(j)}(t_n^{(i)})^{-1}S_z$ meets K frequently. Since $(K_n)_{n\geqslant 1}$ is an increasing sequence of compact subsets of G such that $G = \bigcup_{n\geqslant 1} \operatorname{Int}(K_n)$ there exists N such that $K \subset K_N$. Then there exists $n_0 \geqslant N$ such that $t_{n_0}^{(j)}(t_{n_0}^{(i)})^{-1}S_z$ meets K_{n_0}. But this implies that $t_{n_0}^{(j)} \in K_{n_0} t_{n_0}^{(i)} S_z$ because S_z is normal, contradicting the construction of the sequence $(t_n^{(j)})$. Thus $z, z, z, \ldots$ converges k-times in X/G to z.

(2) $\Rightarrow$ (3). Suppose that (2) holds. Let V be an open neighbourhood of z and $M > 0$. By the continuity of the action on the locally compact Hausdorff space X, there exists an open neighbourhood U of z and a compact neighbourhood K of e in G such that $K \cdot U \subset V$. Then $q_z(K)$ is a compact neighbourhood of the identity in G/S_z and we may choose $k \in \mathbb{P}$ such that $k\nu_z(q_z(K)) > M$. By (2) there exist k sequences $(t_n^{(i)})_{n\geqslant 1}$ such that $t_n^{(i)} \cdot z \to z$ as $n \to \infty$ for each $1 \leqslant i \leqslant k$, and

$$t_n^{(j)}\big(t_n^{(i)}\big)^{-1}S_z \to \infty \quad \text{as } n \to \infty \ (1 \leqslant i < j \leqslant k).$$

Hence there exists n_0 such that $t_{n_0}^{(i)} \cdot z \in U$ for $1 \leqslant i \leqslant k$ and $t_{n_0}^{(j)}(t_{n_0}^{(i)})^{-1}S_z \subset G \setminus (K^{-1}K)$ for $1 \leqslant i < j \leqslant k$. Then $Kt_{n_0}^{(i)} \cdot z \subset K \cdot U \subset V$ and hence $q_z(Kt_{n_0}^{(i)}) \subset q_z(\phi_z^{-1}(V))$ for $1 \leqslant i \leqslant k$. Also $q_z(Kt_{n_0}^{(i)}) \cap q_z(Kt_{n_0}^{(j)}) = \emptyset$ because $Kt_{n_0}^{(i)}S_z \cap Kt_{n_0}^{(j)}S_z = \emptyset$ unless $i = j$. Since the measure ν_z is right-invariant, $\nu_z(q_z(\phi_z^{-1}(V))) \geqslant k\nu_z(q_z(K)) > M$. Since M was arbitrary, (3) follows.

(3) $\Rightarrow$ (4). Compact subsets have finite Haar measure, so this is immediate.

(4) $\Rightarrow$ (1). Suppose that $G \cdot z$ is locally closed in X. Then $G \cdot z$ is a relatively open subset of the locally compact space $\overline{G \cdot z}$ and hence $G \cdot z$ is locally compact. Thus $(G, G \cdot z)$ is a second-countable, locally compact, Hausdorff transformation group. In particular, it follows from [12, Theorem 1] that the map $\psi_z : G/S_z \to G \cdot z : sS_z \mapsto s \cdot z$ is a homeomorphism. Let U be an open subset of X such that $U \cap \overline{G \cdot z} = G \cdot z$. Let N be a compact neighbourhood of z in X such that $N \subset U$. Then $N \cap \overline{G \cdot z} = N \cap G \cdot z$ is a compact subset of $G \cdot z$. Hence $\psi_z^{-1}(N)$ is compact in G/S_z. Now $\psi_z^{-1}(N) = q_z(\phi_z^{-1}(N)) \supset q_z(\phi_z^{-1}(\operatorname{Int} N))$; this contradicts (4) with $V = \operatorname{Int} N$. $\square$

Lemma 3.3 is used in Lemma 3.4 and in Theorem 4.4 below.

Lemma 3.3. *Let (G, X) be a second-countable transformation group. Let $z \in X$ and let $(x_n)_n$ be a sequence in X such that $S_{x_n} \to S_z$ in Σ. Suppose that W is a compact subset of G. Then*

$$\chi_{WS_{x_n}}(r) \to \chi_{WS_z}(r)$$

for $r \in (G \setminus WS_z) \cup (\operatorname{Int} W)S_z$. If $\mu(WS_z \setminus (\operatorname{Int} W)S_z) = 0$, then $\chi_{WS_{x_n}} \to \chi_{WS_z}$ almost everywhere.

Proof. The second statement follows immediately from the first.

Since W is compact both WS_{x_n} and WS_z are closed. We claim that $WS_{x_n} \to WS_z$ in the Fell topology on the closed subsets of G. First let $s \in WS_z$, say $s = wt$ where $w \in W$ and $t \in S_z$. There exist a subsequence (x_{n_i}) and $t_{n_i} \in S_{x_{n_i}}$ such that $t_{n_i} \to t$. So $wt_{n_i} \to wt = s$ and $wt_{n_i} \in WS_{n_i}$ as required. Second, consider (s_n) with $s_n \in WS_{x_n}$, say $s_n = w_n t_n$, and suppose that $s_n \to s$. By compactness there exists a subsequence (w_{n_i}) such that $w_{n_i} \to w \in W$. Then

$t_{n_i} = w_{n_i}^{-1} s_{n_i} \to w^{-1}s$, and hence $w^{-1}s \in S_z$. Thus $s \in wS_z \subset WS_z$ as required, and $WS_{x_n} \to WS_z$ in the Fell topology as claimed.

It follows from $WS_{x_n} \to WS_z$ that $\chi_{WS_{x_n}}(s) \to \chi_{WS_z}(s) = 0$ for $s \in G \setminus WS_z$. For if not then there exists a subsequence x_{n_i} such that $s \in WS_{x_{n_i}}$. Set $s_{n_i} = s$ and note $s_{n_i} \to s$. But then $s \in WS_z$ since $WS_{x_n} \to WS_z$, a contradiction.

Now let $s \in (\operatorname{Int} W)S_z$. We first claim that there exists a subsequence (x_{n_i}) such that $\chi_{WS_{x_{n_i}}}(s) \to \chi_{WS_z}(s) = 1$. To see this, write $s = wt$ where $w \in \operatorname{Int} W$ and $t \in S_z$. Then there exist $t_{n_i} \in S_{x_{n_i}}$ such that $t_{n_i} \to t$. Choose a symmetric neighbourhood U of e in G such that $wU \subseteq \operatorname{Int} W$. Then $t_{n_i} \in Ut$ eventually, so eventually there exist $u_i \in U$ such that $t_{n_i} = u_i t$. Thus $s = wt = wu_i^{-1} t_{n_i} \in wUS_{n_i} \subseteq (\operatorname{Int} W)S_{x_{n_i}}$. So $\chi_{WS_{x_{n_i}}}(s) = 1$ eventually, as claimed.

Now suppose $s \in (\operatorname{Int} W)S_z$ and that $\chi_{WS_{x_n}}(s) \not\to \chi_{WS_z}(s) = 1$. Then there exists a subsequence $(x_{n(j)})$ such that $s \notin WS_{x_{n(j)}}$ for all j. But applying the argument of the preceding paragraph we get a further subsequence $(x_{n(j(i))})$ such that $s \in WS_{x_{n(j(i))}}$ eventually, a contradiction. Thus $\chi_{WS_{x_n}}(s) \to \chi_{WS_z}(s) = 1$ for $s \in (\operatorname{Int} W)S_z$. □

Lemma 3.4. *Let (G, X) be a second-countable transformation group. Let $z \in X$, and let $(x_n)_{n\geqslant 1}$ be a sequence in X such that $S_{x_n} \to S_z$ in Σ, and S_z and all the S_{x_n} are normal in G. Let W be a compact subset of G. Then $\limsup_n \nu_{x_n}(q_{x_n}(W)) \leqslant \nu_z(q_z(W))$.*

Proof. Since $S_{x_n} \to S_z$, $\chi_{WS_{x_n}}(r) \to \chi_{WS_z}(r)$ for $r \in G \setminus WS_z$ by Lemma 3.3. By [23, Proposition 2.18] or [24, Proposition H.17] there exists a "cut-down approximate Bruhat cross section" $b \in C_c(G \times \Sigma)$: thus $b \geqslant 0$ and, for all $H \in \Sigma$, $\int_H b(rt, H)\, d\alpha_H(t) = 1$ for $r \in WH$. Now

$$\begin{aligned}
\nu_{x_n}\big(q_{x_n}(W)\big) &= \int_{G/S_{x_n}} \chi_{q_{x_n}(W)}(\dot r)\, d\nu_{x_n}(\dot r) \\
&= \int_{G/S_{x_n}} \chi_{q_{x_n}(W)}(\dot r) \int_{S_{x_n}} b(rt, S_{x_n})\, d\alpha_{x_n}(t)\, d\nu_{x_n}(\dot r) \\
&= \int_{G/S_{x_n}} \int_{S_{x_n}} \chi_{WS_{x_n}}(rt) b(rt, S_{x_n})\, d\alpha_{x_n}(t)\, d\nu_{x_n}(\dot r) \\
&= \int_G \chi_{WS_{x_n}}(r) b(r, S_{x_n})\, d\mu(r) \quad \big(\text{using (2.2)}\big) \qquad (3.1) \\
&\leqslant \int_{r\in WS_z} b(r, S_{x_n})\, d\mu(r) + \int_{r\in G\setminus WS_z} \chi_{WS_{x_n}}(r) b(r, S_{x_n})\, d\mu(r).
\end{aligned}$$

Let K be the image of $\operatorname{supp} b$ under the coordinate projection $G \times \Sigma \to G$. Since $S_{x_n} \to S_z$, b is continuous, and $\chi_{WS_{x_n}}(r) \to 0$ for $r \in G \setminus WS_z$, we may apply the Dominated Convergence Theorem with dominating functions $\|b\|_\infty \chi_{(WS_z)\cap K}$ and $\|b\|_\infty \chi_{(G\setminus WS_z)\cap K}$ to show that the sum of integrals converges to

$$\int_{r\in WS_z} b(r, S_z)\, d\mu(r) + 0 = \int_G \chi_{WS_z}(r) b(r, S_z)\, d\mu(r).$$

But $\int_G \chi_{WS_z}(r)b(r,S_z)\,d\mu(r)$ equals $\nu_z(q_z(W))$ by the calculation above ending at (3.2). Thus

$$\nu_{x_n}\big(q_{x_n}(W)\big) \leqslant \int\limits_{r\in WS_z} b(r,S_{x_n})\,d\mu(r) + \int\limits_{r\in G\setminus WS_z} \chi_{WS_{x_n}}(r)b(r,S_{x_n})\,d\mu(r) \to \nu_z\big(q_z(W)\big),$$

and the lemma follows. □

We can now extend [3, Proposition 4.1] to the non-free case.

Proposition 3.5. *Let (G,X) be a second-countable transformation group. Let $z\in X$ with $G\cdot z$ locally closed in X and S_z compact. Assume that $(x_n)_{n\geqslant 1}$ is a sequence in X such that $S_{x_n}\to S_z$ in Σ, that S_{x_n} and S_z are normal in G, and that $G\cdot z$ is the unique limit of $(G\cdot x_n)_n$ in X/G. Let $k\in\mathbb{P}$, and suppose that there exists a basic sequence $(W_m)_{m\geqslant 1}$ of compact neighbourhoods of z with $W_{m+1}\subset W_m$, such that, for each m,*

$$\liminf_n \nu_{x_n}\big(q_{x_n}\big(\phi_{x_n}^{-1}(W_m)\big)\big) > (k-1)\nu_z\big(q_z\big(\phi_z^{-1}(W_m)\big)\big).$$

Then (x_n) converges k-times in X/G to z.

Proof. Since $G\cdot z$ is locally closed in X, there is an open subset U of X such that $U\cap\overline{G\cdot z}=G\cdot z$. We may assume, without loss of generality, that $W_m\subset U$ for all $m\geqslant 1$. It then follows from [12, Theorem 1], as in the proof of Lemma 3.2, that $q_z(\phi_z^{-1}(W_m))$ is compact in G/S_z for all $m\geqslant 1$. Let $(K_m)_{m\geqslant 1}$ be an increasing sequence of compact subsets of G such that $G=\bigcup_{m\geqslant 1}\mathrm{Int}(K_m)$.

Let $m\geqslant 1$. It follows from the regularity of ν_z that there exists an open neighbourhood V_m of $q_z(\phi_z^{-1}(W_m))$ and $\epsilon_m>0$ such that

$$\liminf_n \nu_{x_n}\big(q_{x_n}\big(\big(\phi_{x_n}^{-1}(W_m)\big)\big)\big) > (k-1)\nu_z(V_m)+\epsilon_m.$$

Since $q_z(\phi_z^{-1}(W_m))$ is compact and G/S_z is locally compact, there exists an open precompact neighbourhood A_m of $q_z(\phi_z^{-1}(W_m))$ such that $\overline{A_m}\subset V_m$. We have

$$\liminf_n \nu_{x_n}\big(q_{x_n}\big(\big(\phi_{x_n}^{-1}(W_m)\big)\big)\big) > (k-1)\nu_z(\overline{A_m})+\epsilon_m. \tag{3.2}$$

Set $U_m:=q_z^{-1}(A_m)$, so that U_m is an open neighbourhood of $\phi_z^{-1}(W_m)$ and is precompact since both $\overline{A_m}$ and S_z are compact.

We will construct, by induction, a strictly increasing sequence of positive integers $(n_m)_{m\geqslant 1}$ such that, for all $n\geqslant n_m$,

$$\nu_{x_n}\big(q_{x_n}\big(K_m s S_{x_n}\cap\phi_{x_n}^{-1}(W_m)\big)\big) \leqslant \nu_z(\overline{A_m})+\epsilon_m/k \quad \text{for all } s\in\phi_{x_n}^{-1}(W_m), \quad \text{and} \tag{3.3}$$

$$\nu_{x_n}\big(q_{x_n}\big(\phi_{x_n}^{-1}(W_m)\big)\big) > (k-1)\nu_z(\overline{A_m})+\epsilon_m. \tag{3.4}$$

We construct n_1 by applying [3, Lemma 3.2] to K_1, W_1 and U_1 to obtain n_1 such that for every $n \geqslant n_1$ and every $s \in \phi_{x_n}^{-1}(W_1)$ there exists $r \in \phi_z^{-1}(W_1)$ such that $K_1 s \cap \phi_{x_n}^{-1}(W_1) \subset U_1 r^{-1} s$. Then $K_1 s S_{x_n} \cap \phi_{x_n}^{-1}(W_1) \subset U_1 r^{-1} s S_{x_n}$ and hence

$$\nu_{x_n}\big(q_{x_n}\big(K_1 s S_{x_n} \cap \phi_{x_n}^{-1}(W_1)\big)\big) \leqslant \nu_{x_n}\big(q_{x_n}\big(U_1 r^{-1} s S_{x_n}\big)\big) = \nu_{x_n}\big(q_{x_n}(U_1)\big)$$

because S_{x_n} is normal and the measures ν_{x_n} have been chosen to be right invariant. Since $S_{x_n} \to S_z$, $\limsup \nu_{x_n}(q_{x_n}(\overline{U_1})) \leqslant \nu_z(q_z(\overline{U_1})) \leqslant \nu_z(\overline{A_1})$ by Lemma 3.4. So by increasing n_1 if necessary, (3.3) holds for $m = 1$. If necessary, we can increase n_1 again to ensure that (3.4) holds by using (3.2) with $m = 1$.

Assuming that we have constructed $n_1 < n_2 < \cdots < n_{m-1}$, we apply [3, Lemma 3.2] to K_m, W_m and U_m to obtain $n_m > n_{m-1}$ such that for $n > n_m$ and every $s \in \phi_{x_n}^{-1}(W_m)$ there exists $r \in \phi_z^{-1}(W_1)$ such that $K_m s \cap \phi_{x_n}^{-1}(W_m) \subset U_m r^{-1} s$. Then $K_m s S_{x_n} \cap \phi_{x_n}^{-1}(W_m) \subset U_m r^{-1} s S_{x_n}$. Applying Lemma 3.4 to $\overline{U_m}$, we have

$$\nu_{x_n}\big(q_{x_n}\big(K_m s S_{x_n} \cap \phi_{x_n}^{-1}(W_m)\big)\big) \leqslant \nu_{x_n}\big(q_{x_n}(U_m)\big) \leqslant \nu_{x_n}\big(q_{x_n}(\overline{U_m})\big) \leqslant \nu_z\big(q_z(\overline{U_m})\big) + \epsilon_m/k$$

eventually. Since $q_z(\overline{U_m}) \subset A_m$ we can increase n_m twice if necessary to obtain first (3.3) and then (3.4).

If $n_1 > 1$ then, for $1 \leqslant n < n_1$, we set $t_n^{(i)} = e$ for $1 \leqslant i \leqslant k$. For each $n \geqslant n_1$ there is a unique m such that $n_m \leqslant n < n_{m+1}$. Choose $t_n^{(1)} \in \phi_{x_n}^{-1}(W_m)$. Using (3.3) and (3.4)

$$\begin{aligned} \nu_{x_n}\big(q_{x_n}\big(\phi_{x_n}^{-1}(W_m) \setminus K_m t_n^{(1)} S_{x_n}\big)\big) &\geqslant \nu_{x_n}\big(q_{x_n}\big(\phi_{x_n}^{-1}(W_m)\big) \setminus q_{x_n}\big(\phi_{x_n}^{-1}(W_m) \cap K_m t_n^{(1)} S_{x_n}\big)\big) \\ &> (k-2)\nu_z(\overline{A_m}) + (k-1)\epsilon_m/k. \end{aligned} \tag{3.5}$$

So if $k \geqslant 2$ we may choose $t_n^{(2)}$ such that $t_n^{(2)} \in \phi_{x_n}^{-1}(W_m) \setminus K_m t_n^{(1)} S_{x_n}$.

Next, using the formal relation $q_{x_n}(A \setminus (B \cup C)) \supset q_{x_n}(A \setminus B) \setminus q_{x_n}(A \cap C)$, and (3.3) and (3.5)

$$\nu_{x_n}\big(q_{x_n}\big(\phi_{x_n}^{-1}(W_m) \setminus \big(K_m t_n^{(1)} S_{x_n} \cup K_m t_n^{(2)} S_{x_n}\big)\big)\big) > (k-3)\nu_z(\overline{A_m}) + (k-2)\epsilon_m/k.$$

So if $k \geqslant 3$ we may choose $t_n^{(3)} \in \phi_{x_n}^{-1}(W_m) \setminus (K_m t_n^{(1)} S_{x_n} \cup K_m t_n^{(2)} S_{x_n})$. Continuing in this way, we obtain $t_n^{(1)}, \dots, t_n^{(k)} \in \phi_{x_n}^{-1}(W_m)$ such that, for $1 \leqslant j \leqslant k$,

$$t_n^{(j)} \in \phi_{x_n}^{-1}(W_m) \setminus \left(\bigcup_{i=1}^{j-1} K_m t_n^{(i)} S_{x_n} \right).$$

Note that for $n_m \leqslant n < n_{m+1}$ we have

$$t_n^{(i)} \cdot x_n \in W_m \quad \text{for } 1 \leqslant i \leqslant k \quad \text{and} \quad t_n^{(j)} \notin K_m t_n^{(i)} S_{x_n} \quad \text{for } 1 \leqslant i < j \leqslant k.$$

Now we will show that x_n converges k-times in X/G to z. To see that $t_n^{(i)} \cdot x_n \to z$ as $n \to \infty$ for $1 \leqslant i \leqslant k$, fix i and let V be a neighbourhood of z. There exists m_0 such that $W_m \subset V$

for all $m \geqslant m_0$. For each $n \geqslant n_{m_0}$ there exists $m \geqslant m_0$ such that $n_m \leqslant n < n_{m+1}$, and thus $t_n^{(i)} \cdot x_n \in W_m \subset V$. Thus $t_n^{(i)} \cdot x_n \to z$ as $n \to \infty$ for $1 \leqslant i \leqslant k$ as claimed.

Finally, fix a compact set K in G. There exists m_0 such that $K \subset K_m$ for all $m \geqslant m_0$. Then for $m \geqslant m_0$, $n_m \leqslant n < n_{m+1}$ and $1 \leqslant i < j \leqslant k$ we have $t_n^{(j)} \notin K_m t_n^{(i)} S_{x_n} = K_m S_{x_n} t_n^{(i)}$ and hence $t_n^{(j)} (t_n^{(i)})^{-1} S_{x_n} \subset G \setminus K$. So for $n \geqslant n_{m_0}$ and $1 \leqslant i < j \leqslant k$, $t_n^{(j)} (t_n^{(i)})^{-1} S_{x_n} \subset G \setminus K$ as required. □

4. k-Times convergence and lower bounds on multiplicity

Throughout this section G is assumed to be abelian. Let $k \in \mathbb{P}$. Here we introduce induced representations of the crossed product $C_0(X) \rtimes G$, and consider a sequence $(\pi_n)_n$ of induced representations converging to an induced representation π. We establish sufficient conditions which ensure that $M_L(\pi, (\pi_n)) \geqslant k$. The dual action of the character group $\hat{G}$ on $C_0(X) \rtimes G$ plays a major role in our approach.

We start with some background on induced representations. If $\tau \in \hat{G}$ then "the representation $\mathrm{Ind}_{x,S_x}^G(\tau|)$ of $C_0(X) \rtimes G$ induced from $\tau|$ on S_x" is irreducible by [22, Proposition 4.2]. By [22, Lemma 4.14] $\mathrm{Ind}_{x,S_x}^G(\tau|)$ is unitarily equivalent to $\mathrm{Ind}(x,\tau) := M_x \rtimes V_\tau$ on $L^2(G/S_x, \nu_x)$, where

$$\big(M_x(f)\xi\big)(\dot{s}) = f(s \cdot x)\xi(\dot{s}) \quad \text{and} \quad \big(V_\tau(t)\xi\big)(\dot{s}) = \tau(t)\xi\big(t^{-1} \cdot \dot{s}\big)$$

for $f \in C_0(X)$ and $\xi \in L^2(G/S_x)$.

Let $x, y \in X$ and $\tau, \sigma \in \hat{G}$. Write $(x,\tau) \sim (y,\sigma)$ if and only if $\overline{G \cdot x} = \overline{G \cdot y}$ and $\tau|_{S_x} = \sigma|_{S_x}$. Then $\sim$ is an equivalence relation. Since G is abelian, Theorem 5.3 of [22] implies that the map $(x,\tau) \mapsto \ker \mathrm{Ind}(x,\tau)$ induces a homeomorphism of $(X \times \hat{G})/{\sim}$ onto the primitive ideal space of $C_0(X) \rtimes G$. The proof of [22, Theorem 5.3] also shows that the quotient map $X \times \hat{G} \to (X \times \hat{G})/{\sim}$ is open.

Let $\hat{\alpha}$ be the dual action of $\hat{G}$ on $C_0(X) \rtimes G$, that is,

$$\hat{\alpha}_\tau(b)(s) = \tau(s)b(s) \quad \text{for } b \in C_c\big(G, C_0(X)\big) \text{ and } \tau \in \hat{G}.$$

An easy calculation shows that $\mathrm{Ind}(x,\tau) = \mathrm{Ind}(x,1) \circ \hat{\alpha}_\tau$ [16, Lemma 5.5]. The next two lemmas will be needed for the proof of Proposition 4.3 where we will show that if $\mathrm{Ind}(x_n,\tau_n) \to \mathrm{Ind}(z,\tau)$ then $M_L(\mathrm{Ind}(z,\tau), (\mathrm{Ind}(x_n,\tau_n)))$ does not depend on (τ_n) or τ.

Lemma 4.1. *Let (G,X) be a second-countable transformation group with G abelian. Suppose that a net $(\mathrm{Ind}(x_\lambda,\tau_\lambda))_{\lambda \in \Lambda}$ converges to $\mathrm{Ind}(z,\tau)$ in the spectrum of $C_0(X) \rtimes G$. Then there exists a subnet $(\mathrm{Ind}(x_{\lambda(\gamma)},\tau_{\lambda(\gamma)}))_{\gamma \in \Gamma}$ and a net $(\sigma_\gamma)_{\gamma \in \Gamma}$ in $\hat{G}$ such that $\sigma_\gamma \to \tau$ in $\hat{G}$, $\mathrm{Ind}(x_{\lambda(\gamma)},\tau_{\lambda(\gamma)}) \simeq \mathrm{Ind}(x_{\lambda(\gamma)},\sigma_\gamma)$ and*

$$M_L\big(\mathrm{Ind}(z,\tau), \big(\mathrm{Ind}(x_\lambda,\tau_\lambda)\big)\big) = M_L\big(\mathrm{Ind}(z,\tau), \big(\mathrm{Ind}(x_{\lambda(\gamma)},\sigma_\gamma)\big)\big).$$

Proof. By [5, Proposition 2.3] there exists a subnet $(\mathrm{Ind}(x_{\lambda(\delta)},\tau_{\lambda(\delta)}))_{\delta \in \Delta}$ such that

$$\begin{aligned} M_L\big(\mathrm{Ind}(z,\tau), \big(\mathrm{Ind}(x_\lambda,\tau_\lambda)\big)\big) &= M_U\big(\mathrm{Ind}(z,\tau), \big(\mathrm{Ind}(x_{\lambda(\delta)},\tau_{\lambda(\delta)})\big)\big) \\ &= M_L\big(\mathrm{Ind}(z,\tau), \big(\mathrm{Ind}(x_{\lambda(\delta)},\tau_{\lambda(\delta)})\big)\big). \end{aligned} \tag{4.1}$$

Since the quotient map $X \times \hat{G} \to (X \times \hat{G})/\sim$ is open, there exist a subnet $(x_{\lambda(\delta(\gamma))}, \tau_{\lambda(\delta(\gamma))})_{\gamma \in \Gamma}$ of $(x_{\lambda(\delta)}, \tau_{\lambda(\delta)})$ and a net $(y_\gamma, \sigma_\gamma)$ in $X \times \hat{G}$ such that $(x_{\lambda(\delta(\gamma))}, \tau_{\lambda(\delta(\gamma))}) \sim (y_\gamma, \sigma_\gamma) \to (z, \tau)$. Then $(x_{\lambda(\delta(\gamma))}, \tau_{\lambda(\delta(\gamma))}) \sim (x_{\lambda(\delta(\gamma))}, \sigma_\gamma)$ and so $\mathrm{Ind}(x_{\lambda(\delta(\gamma))}, \tau_{\lambda(\delta(\gamma))}) = \mathrm{Ind}(x_{\lambda(\delta(\gamma))}, \sigma_\gamma)$. By (4.1), passing to the subnet $(\mathrm{Ind}(x_{\lambda(\delta(\gamma))}, \tau_{\lambda(\delta(\gamma))}))$ does not change the relative M_L. □

Lemma 4.2. *Let (G, X) be a second-countable transformation group with G abelian. Suppose that $(x_\lambda)_{\lambda \in \Lambda}$ and $(\tau_\lambda)_{\lambda \in \Lambda}$ are nets in X and $\hat{G}$ such that $\tau_\lambda \to 1$. Let $z \in X$ and ξ a unit vector in $L^2(G/S_z)$ and $k \in \mathbb{P}$. For each λ, let $\{\xi_\lambda^{(i)}\colon 1 \leqslant i \leqslant k\}$ be an orthonormal set in $L^2(G/S_z)$. For $1 \leqslant i \leqslant k$,*

$$\big\langle \mathrm{Ind}(x_\lambda, 1)(\cdot)\xi_\lambda^{(i)}, \xi_\lambda^{(i)}\big\rangle \to_\lambda \big\langle \mathrm{Ind}(z, 1)(\cdot)\xi, \xi\big\rangle \tag{4.2}$$

if and only if

$$\big\langle \mathrm{Ind}(x_\lambda, \tau_\lambda)(\cdot)\xi_\lambda^{(i)}, \xi_\lambda^{(i)}\big\rangle \to_\lambda \big\langle \mathrm{Ind}(z, 1)(\cdot)\xi, \xi\big\rangle. \tag{4.3}$$

Proof. Assume (4.2) holds and let $b \in C_0(X) \rtimes G$. Let $\hat{\alpha}$ be the dual action. Since $\mathrm{Ind}(x_\lambda, \tau_\lambda)(b) = \mathrm{Ind}(x_\lambda, 1) \circ \hat{\alpha}_{\tau_\lambda}(b)$ and $\|\mathrm{Ind}(x_\lambda, 1)(b - \hat{\alpha}_{\tau_\lambda}(b))\| \leqslant \|b - \hat{\alpha}_{\tau_\lambda}(b)\| \to 0$, (4.3) holds. Similarly, if (4.3) holds then so does (4.2) using $\|b - \hat{\alpha}_{\tau_\lambda^{-1}}(b)\| \to 0$. □

Recall that if A is a C^*-algebra, $\pi \in \hat{A}$ and Ω is a net in $\hat{A}$, then $M_L(\pi, \Omega) > 0$ if and only if Ω converges to π [5, p. 206].

Proposition 4.3. *Let (G, X) be a second-countable transformation group with G abelian. Suppose that* $\mathrm{Ind}(x_n, \tau_n) \to \mathrm{Ind}(z, \tau)$. *Then*

$$\begin{aligned} M_L\big(\mathrm{Ind}(z, \tau), \big(\mathrm{Ind}(x_n, \tau_n)\big)\big) &= M_L\big(\mathrm{Ind}(z, 1), \big(\mathrm{Ind}\big(x_n, \tau_n\tau^{-1}\big)\big)\big) \\ &= M_L\big(\mathrm{Ind}(z, 1), \big(\mathrm{Ind}(x_n, 1)\big)\big) \end{aligned}$$

and

$$\begin{aligned} M_U\big(\mathrm{Ind}(z, \tau), \big(\mathrm{Ind}(x_n, \tau_n)\big)\big) &= M_U\big(\mathrm{Ind}(z, 1), \big(\mathrm{Ind}\big(x_n, \tau_n\tau^{-1}\big)\big)\big) \\ &= M_U\big(\mathrm{Ind}(z, 1), \big(\mathrm{Ind}(x_n, 1)\big)\big). \end{aligned}$$

Proof. We have

$$M_L\big(\mathrm{Ind}(z, \tau), \big(\mathrm{Ind}(x_n, \tau_n)\big)\big) = M_L\big(\mathrm{Ind}(z, 1) \circ \hat{\alpha}_\tau, \big(\mathrm{Ind}\big(x_n, \tau_n\tau^{-1}\big) \circ \hat{\alpha}_\tau\big)\big).$$

Since $\hat{\alpha}_\tau$ is an automorphism of $C_0(X) \rtimes G$, we obtain the first equality.

To show the second equality, fix a pure state ϕ associated with $\mathrm{Ind}(z, 1)$. Let $\sigma_n = \tau_n\tau^{-1}$. We will use the criterion of [6, Lemma 5.2(ii)] to prove that

$$M_L\big(\mathrm{Ind}(z, 1), \big(\mathrm{Ind}(x_n, \sigma_n)\big)\big) = M_L\big(\mathrm{Ind}(z, 1), \big(\mathrm{Ind}(x_n, 1)\big)\big). \tag{4.4}$$

First, suppose that $M_L(\mathrm{Ind}(z, 1), (\mathrm{Ind}(x_n, 1))) \geqslant k$ for some $k \in \mathbb{P}$. We will show that $M_L(\mathrm{Ind}(z, 1), (\mathrm{Ind}(x_n, \sigma_n))) \geqslant k$ as well. Since $\mathrm{Ind}(x_n, 1) \to \mathrm{Ind}(z, 1)$ we have $k \geqslant 1$. Let

$\Omega = (\mathrm{Ind}(x_{n_\lambda}, \sigma_{n_\lambda}))_{\lambda \in \Lambda}$ be any subnet of $(\mathrm{Ind}(x_n, \sigma_n))_n$. By passing to a further subnet and relabelling we may assume that $\sigma_{n_\lambda} \to 1$ in $\hat{G}$ (see Lemma 4.1).

Consider the subnet $\Omega' := (\mathrm{Ind}(x_{n_\lambda}, 1))_{\lambda \in \Lambda}$ of $(\mathrm{Ind}(x_n, 1))_n$. By [6, Lemma 5.2] there exists a further subnet $(\mathrm{Ind}(x_{n_{\lambda(\mu)}}, 1))_{\mu \in \Upsilon}$ which has the k-vector property for ϕ. So by Lemma 4.2, $(\mathrm{Ind}(x_{n_{\lambda(\mu)}}, \sigma_{n_{\lambda(\mu)}}))_{\mu \in \Upsilon}$ has the k-vector property as well and hence $M_L(\mathrm{Ind}(z, 1), (\mathrm{Ind}(x_n, \sigma_n))) \geqslant k$. It follows that $M_L(\mathrm{Ind}(z, 1), (\mathrm{Ind}(x_n, \sigma_n))) \geqslant M_L(\mathrm{Ind}(z, 1), (\mathrm{Ind}(x_n, 1)))$.

Second, note that $\mathrm{Ind}(x_n, \sigma_n) \to \mathrm{Ind}(z, 1)$. Suppose that $M_L(\mathrm{Ind}(z, 1), (\mathrm{Ind}(x_n, \sigma_n))) \geqslant k$ for some $k \geqslant 1$. Let $\Omega = (\mathrm{Ind}(x_{n_\lambda}, 1))_{\lambda \in \Lambda}$ be any subnet of $(\mathrm{Ind}(x_n, 1))_n$. Consider $\Omega' := (\mathrm{Ind}(x_{n_\lambda}, \sigma_{n_\lambda}))_{\lambda \in \Lambda}$; by passing to a further subnet and relabelling we may assume that $\sigma_{n_\lambda} \to 1$. By [6, Lemma 5.2], there exists a subnet $(\mathrm{Ind}(x_{n_{\lambda(\mu)}}, \sigma_{n_{\lambda(\mu)}}))_{\mu \in \Upsilon}$ of Ω' with the k-vector property for ϕ. So by Lemma 4.2, $(\mathrm{Ind}(x_{n_{\lambda(\mu)}}, 1))_{\mu \in \Upsilon}$ has the k-vector property as well and hence $M_L(\mathrm{Ind}(z, 1), (\mathrm{Ind}(x_n, 1))) \geqslant k$. Thus (4.4) holds.

The proof of the corresponding statement about the upper multiplicities is similar, using the criterion of [6, Lemma 5.2(i)]. $\square$

The next theorem is an analogue of [2, Theorem 2.3] and [4, Theorem 2.1]. In Corollary 4.6, we will show that the three hypotheses of Theorem 4.4 can be satisfied under a suitable assumption of k-times convergence.

Theorem 4.4. *Let (G, X) be a second-countable transformation group with G abelian. Let $k \in \mathbb{P}$, $z \in X$. Let $(x_n)_n$ be a sequence in X such that $S_{x_n} \to S_z$. Suppose that there exists a compact, symmetric neighbourhood W of the identity in G and k sequences $(t_n^{(1)})_n, (t_n^{(2)})_n, \ldots, (t_n^{(k)})_n$ in G such that*

(1) $t_n^{(j)} \cdot x_n \to z$ *for* $1 \leqslant j \leqslant k$;
(2) *there exists N such that $n \geqslant N$ implies* $Wt_n^{(j)} S_{x_n} \cap Wt_n^{(i)} S_{x_n} = \emptyset$ *for* $1 \leqslant i < j \leqslant k$;
(3) $\mu(WS_z \setminus (\mathrm{Int}\, W)S_z) = 0$.

If $\mathrm{Ind}(x_n, \tau_n) \to \mathrm{Ind}(z, \tau)$ *as $n \to \infty$ then* $M_L(\mathrm{Ind}(z, \tau), (\mathrm{Ind}(x_n, \tau_n))) \geqslant k$.

Proof. We will prove that $M_L(\mathrm{Ind}(z, 1), (\mathrm{Ind}(x_n, 1))) \geqslant k$; since G is abelian this suffices by Proposition 4.3.

For $x \in X$, recall that $q_x : G \to G/S_x$ is the quotient map. For $s \in G$ set

$$\eta\big(q_z(s)\big) := \nu_z\big(q_z(WS_z)\big)^{-1/2} \chi_{q_z(WS_z)}\big(q_z(s)\big);$$

for $n \geqslant 1$ and $1 \leqslant j \leqslant k$ set

$$\eta_n^{(j)}\big(q_{x_n}(s)\big) := \nu_{x_n}\big(q_{x_n}(WS_{x_n})\big)^{-1/2} \chi_{q_{x_n}(WS_{x_n})}\big(q_{x_n}\big(st_n^{(j)-1}\big)\big).$$

Then η is a unit vector in $L^2(G/S_z, \nu_z)$, and, for each $n \geqslant 1$ and $1 \leqslant j \leqslant k$, $\eta_n^{(j)}$ is a unit vector in $L^2(G/S_{x_n}, \nu_{x_n})$. Note that $st_n^{(j)-1} \in WS_{x_n}$ and $st_n^{(i)-1} \in WS_{x_n}$ if and only if $s \in Wt_n^{(j)} S_{x_n} \cap Wt_n^{(i)} S_{x_n}$. But if $n \geqslant N$ and $i \neq j$ then $Wt_n^{(j)} S_{x_n} \cap Wt_n^{(i)} S_{x_n} = \emptyset$ by hypothesis (2), and hence $\langle \eta_n^{(j)}, \eta_n^{(i)} \rangle = 0$.

Fix $f \in C_c(G \times X) \subset C_0(X) \rtimes G$ of the form $f(s, x) = h(s)g(x)$ where $h \in C_c(G)$ and $g \in C_c(X)$. We will compute

$$\Psi_n^{(j)}(f) := \langle (\mathrm{Ind}(x_n, 1) f)\eta_n^{(j)}, \eta_n^{(j)} \rangle$$

for $1 \leqslant j \leqslant k$. To simplify the formulas, we write q_n for q_{x_n}, χ_n for $\chi_{q_n(WS_{x_n})}$ and C_n for $\nu_{x_n}(q_n(WS_{x_n}))^{-1}$. We compute using the formulas from [16, p. 1216]:

$$\begin{aligned}
\Psi_n^{(j)}(f) &= \int_{G/S_{x_n}} \big((\mathrm{Ind}(x_n, 1) f)\eta_n^{(j)}\big)(\dot v)\overline{\eta_n^{(j)}(\dot v)}\, d\nu_{x_n}(\dot v) \\
&= C_n \int_{G/S_{x_n}} \int_G h\big(vu^{-1}\big) g(v \cdot x_n)\chi_n\big(q_n\big(ut_n^{(j)-1}\big)\big)\, d\mu(u)\chi_n\big(q_n\big(vt_n^{(j)-1}\big)\big)\, d\nu_{x_n}(\dot v)
\end{aligned}$$

which, via changes of variables $s = ut_n^{(j)-1}$ and $\dot r = q_n(vt_n^{(j)-1})$, is

$$\begin{aligned}
&= C_n \int_{G/S_{x_n}} \int_G h\big(rs^{-1}\big) g\big(rt_n^{(j)} \cdot x_n\big)\chi_n(\dot s)\chi_n(\dot r)\, d\mu(s)\, d\nu_{x_n}(\dot r) \\
&= C_n \int_{G/S_{x_n}} \int_G h(u) g\big(rt_n^{(j)} \cdot x_n\big)\chi_n\big(q_n\big(u^{-1}r\big)\big)\chi_n(\dot r)\, d\mu(u)\, d\nu_{x_n}(\dot r) \\
&= C_n \int_{\dot r \in q_n(WS_{x_n})} g\big(rt_n^{(j)} \cdot x_n\big)\bigg(\int_{u \in rWS_{x_n}} h(u)\, d\mu(u)\bigg)\, d\nu_{x_n}(\dot r).
\end{aligned}$$

For each $n \geqslant 1$,

$$F_n^j(r) = g\big(rt_n^{(j)} \cdot x_n\big) \int_{u \in rWS_{x_n}} h(u)\, d\mu(u)$$

is the formula for a function F_n^j which is constant on cosets of S_{x_n}, and the above calculation shows

$$\Psi_n^{(j)}(f) = C_n \int_{\dot r \in q_n(WS_{x_n})} F_n^j(r)\, d\nu_{x_n}(\dot r). \tag{4.5}$$

Note, for later use, that the F_n^j are uniformly bounded by $\|g\|_\infty \|h\|_1$.

Since $S_{x_n} \to S_z$, it follows from Lemma 3.3 and hypothesis (3) that $\chi_{WS_{x_n}} \to \chi_{WS_z}$ almost everywhere. By the invariance of μ, for each $r \in G$ we have $\chi_{rWS_{x_n}} \to \chi_{rWS_z}$ almost everywhere. By hypothesis (1) and the continuity of g, for all $1 \leqslant j \leqslant k$ and $r \in G$, $g(rt_n^{(j)} \cdot x_n) \to g(r \cdot z)$ as $n \to \infty$. Since $h \in L^1(G, \mu)$ it follows that, for all $r \in G$,

$$F_n^j(r) \to g(r \cdot z) \int_{u \in rWS_z} h(u)\, d\mu(u) \tag{4.6}$$

as $n \to \infty$.

By [23, Proposition 2.18(ii)] or [24, Proposition H.17(a)] we can choose a "cut-down generalised Bruhat approximate cross-section" $b \in C_c(G \times \Sigma)$ such that

$$\int_H b(rt, H)\, d\alpha_H(t) = 1$$

for $r \in WH$. Thus

$$\begin{aligned}
(4.5) &= C_n \int_{G/S_{x_n}} \chi_{WS_{x_n}}(r) F_n^j(r)\, d\nu_{x_n}(\dot{r}) \\
&= C_n \int_{G/S_{x_n}} \chi_{WS_{x_n}}(r) F_n^j(r) \int_{S_{x_n}} b(rt, S_{x_n})\, d\alpha_{x_n}(t)\, d\nu_{x_n}(\dot{r}) \\
&= C_n \int_{G/S_{x_n}} \int_{S_{x_n}} \chi_{WS_{x_n}}(rt) F_n^j(rt) b(rt, S_{x_n})\, d\alpha_{x_n}(t)\, d\nu_{x_n}(\dot{r}) \\
&= C_n \int_G \chi_{WS_{x_n}}(r) F_n^j(r) b(r, S_{x_n})\, d\mu(r)
\end{aligned}$$

using (2.2). Similarly,

$$\begin{aligned}
C_n^{-1} &= \nu_{x_n}\big(q_n(WS_{x_n})\big) \\
&= \int_{G/S_{x_n}} \chi_{WS_{x_n}}(r) \int_{S_{x_n}} b(rt, S_{x_n})\, d\alpha_{x_n}(t)\, d\nu_{x_n}(\dot{r}) \\
&= \int_{G/S_{x_n}} \int_{S_{x_n}} \chi_{WS_{x_n}}(rt) b(rt, S_{x_n})\, d\alpha_{x_n}(t)\, d\nu_{x_n}(\dot{r}) \\
&= \int_G \chi_{WS_{x_n}}(r) b(r, S_{x_n})\, d\mu(r).
\end{aligned}$$

So we have shown that $\Psi_n^{(j)}(f)$ is the product of

$$C_n = \left(\int_G \chi_{WS_{x_n}}(r) b(r, S_{x_n})\, d\mu(r) \right)^{-1} \tag{4.7}$$

and

$$\int_G \chi_{WS_{x_n}}(r) F_n^j(r) b(r, S_{x_n})\, d\mu(r). \tag{4.8}$$

An almost identical calculation shows that

$$\Psi(f) := \big\langle \big(\mathrm{Ind}(z,1)f\big)\eta, \eta \big\rangle$$

is the product of

$$C := \left(\int\limits_G \chi_{WS_z}(r) b(r, S_z)\, d\mu(r) \right)^{-1} \tag{4.9}$$

and

$$\int\limits_G \left(\chi_{WS_z}(r) g(r \cdot z) b(r, S_z) \int\limits_{u \in rWS_z} h(u)\, d\mu(u) \right) d\mu(r). \tag{4.10}$$

We have noted above that $\chi_{WS_{x_n}} \to \chi_{WS_z}$ almost everywhere. Since b is continuous on $G \times \Sigma$, $b(\cdot, S_{x_n}) \to b(\cdot, S_z)$. So the integrand in (4.7) converges pointwise almost everywhere to the integrand in (4.9), and hence C_n converges to C by the Dominated Convergence Theorem. (For an L^1-dominant, let L be the compact subset of G obtained by projecting the support of b and then take $\|b\|_\infty \chi_L$.) Using (4.6), the integrand in (4.8) converges pointwise almost everywhere to the integrand in (4.10), and hence it follows from the Dominated Convergence Theorem that $\Psi_n^{(j)}(f) \to \Psi(f)$ for $1 \leqslant j \leqslant k$. (For an L^1-dominant, we may take $\|g\|_\infty \|h\|_1 \|b\|_\infty \chi_L$.) Since the linear span of such f is norm-dense in $C_0(X) \rtimes G$ and the $\Psi_n^{(j)}$ and Ψ have norm one, it follows that $\Psi_n^{(j)} \to \Psi$ in the weak*-topology for $1 \leqslant j \leqslant k$.

Suppose that $(\mathrm{Ind}(\epsilon_{x_{n_\lambda}}, 1))_{\lambda \in \Lambda}$ is a subnet of $(\mathrm{Ind}(x_n, 1))$. Then there exists $\lambda_0 \in \Lambda$ such that $n_\lambda \geqslant N$ whenever $\lambda \geqslant \lambda_0$. So the calculations above give, for each $\lambda \geqslant \lambda_0$, k mutually orthogonal pure states $\Psi_{n_\lambda}^1, \ldots, \Psi_{n_\lambda}^k$ associated with $\mathrm{Ind}(\epsilon_{x_{n_\lambda}}, 1)$ such that $\lim_\lambda \Psi_{n_\lambda}^j = \Psi$ for $1 \leqslant j \leqslant k$. It now follows from [6, Lemma 5.2(ii)] that $M_L(\mathrm{Ind}(z,1), (\mathrm{Ind}(x_n, 1))) \geqslant k$. $\square$

Before moving to Corollary 4.6, we need to consider the measure-theoretic hypothesis (3) in Theorem 4.4.

Lemma 4.5. *Let G be a locally compact, abelian group with Haar measure μ and let S be a closed subgroup of G such that S is either countable or compact. Then there exists a compact symmetric neighbourhood W of the identity in G such that $\mu(WS \setminus (\mathrm{Int}\, W)S) = 0$.*

Proof. We may assume that $G = H \times \mathbb{R}^k$, for some $k \geqslant 0$, where H is a locally compact abelian group containing a compact open subgroup K [7, Theorem 4.2.1]. If $k = 0$, we may simply take $W = K$. Assuming that $k \geqslant 1$, let $W = K \times B$ where B is the closed unit ball of $\mathbb{R}^k$. Then $\mathrm{Int}\, W = K \times \mathrm{Int}\, B$ and $W \setminus \mathrm{Int}\, W = K \times (B \setminus \mathrm{Int}\, B)$ which has μ-measure zero since the Lebesgue measure of $B \setminus \mathrm{Int}\, B$ is zero.

Suppose that S is countable. Then $WS \setminus (\mathrm{Int}\, W)S \subseteq (W \setminus (\mathrm{Int}\, W))S$ and the right-hand side has μ-measure zero by the invariance of μ and the countability of S.

Suppose, instead, that S is compact. Then the image of S under the projection $p : G = H \times \mathbb{R}^k \to \mathbb{R}^k$ is a compact subgroup of $\mathbb{R}^k$ and hence is $\{0\}$. Thus $S = T \times \{0\}$ where T is a subgroup of H. Then $WS \setminus (\mathrm{Int}\, W)S = KT \times (B \setminus \mathrm{Int}\, B)$ which has μ-measure zero because $B \setminus \mathrm{Int}\, B$ has zero Lebesgue measure. $\square$

A more complicated argument shows that if S is a closed subgroup of a second-countable, locally compact, abelian group $G = H \times \mathbb{R}^k$ such that the image $p(S)$ of the projection of S into $\mathbb{R}^k$ is closed, then there exists a compact symmetric neighbourhood W of the identity in G such that $\mu(WS \setminus (\operatorname{Int} W)S) = 0$. We omit the details as we will not use that result here.

In several places we assume that S_z is compact (for example in Proposition 3.5 and Sections 5 and 6), and so we restrict to this case in the following result.

Corollary 4.6. *Let (G, X) be a second-countable transformation group with G abelian. Let $z \in X$ with S_z compact. Let $k \in \mathbb{P}$ and let $(x_n)_n$ be a sequence in X converging k-times in X/G to z such that $S_{x_n} \to S_z$. If $\operatorname{Ind}(x_n, \tau_n) \to \operatorname{Ind}(z, \tau)$ as $n \to \infty$ then $M_L(\operatorname{Ind}(z, \tau), (\operatorname{Ind}(x_n, \tau_n))) \geqslant k$.*

Proof. Since S_z is compact, by Lemma 4.5 there exists a compact symmetric neighbourhood W of the identity in G such that $\mu(WS_z \setminus (\operatorname{Int} W)S_z) = 0$.

By the k-times convergence there exist k sequences $(t_n^{(1)})_n, (t_n^{(2)})_n, \ldots, (t_n^{(k)})_n$ in G such that $t_n^{(i)} \cdot x_n \to z$ as $n \to \infty$ for $1 \leqslant i \leqslant k$ and $t_n^{(j)}(t_n^{(i)})^{-1} S_{x_n} \to \infty$ as $n \to \infty$ for $i \neq j$. Since W^2 is compact, there exists $N \in \mathbb{P}$ such that $n \geqslant N$ implies $t_n^{(j)} t_n^{(i)-1} S_{x_n} \cap W^2 = \emptyset$. Thus $W t_n^{(j)} S_{x_n} \cap W t_n^{(i)} S_{x_n} = \emptyset$ for $1 \leqslant i < j \leqslant k$ when $n \geqslant N$. The result now follows from Theorem 4.4. $\square$

Next we use well-known transformation groups to give examples of transformation groups with a sequence $x_n \to z$ where the stability subgroup at z is not compact (so that Corollary 4.6 does not apply), but where we can still verify that all the hypotheses of Theorem 4.4 hold.

Examples 4.7. Let $(\mathbb{R}, Y)$ be Green's free non-proper transformation group [14, pp. 95–96]: the space Y is a closed subset of $\mathbb{R}^3$ and consists of countably many orbits, with orbit representatives $y_0 = \mathbf{0} := (0, 0, 0)$ and $y_n = (2^{-2n}, 0, 0)$ for $n = 1, 2, \ldots$. The action of $\mathbb{R}$ is given by $s \cdot y_0 = (0, s, 0)$ for all s; and for $n \geqslant 1$,

$$s \cdot y_n = \begin{cases} (2^{-2n}, s, 0) & \text{if } s \leqslant n; \\ (2^{-2n} - (\frac{s-n}{\pi})2^{-2n-1}, n\cos(s-n), n\sin(s-n)) & \text{if } n < s < n + \pi; \\ (2^{-2n-1}, s - \pi - 2n, 0) & \text{if } s \geqslant n + \pi. \end{cases}$$

Green's action is free and each orbit consists of two vertical lines joined by an arc of a helix situated on a cylinder of radius n; the action moves y_n along the vertical lines at unit speed, and along the arc at radial speed.

Next let $(\mathbb{R}, \mathbb{C})$ be the non-free transformation group of [23, Example 5.4]. Here $\mathbb{R}$ acts on $\mathbb{C}$ by fixing the origin, and if $w \neq 0$ then $r \cdot w = e^{2\pi i (\frac{r}{|w|})} w$. The orbits are concentric circles about the origin and the stability subgroups are

$$S_w = \begin{cases} \mathbb{R} & \text{if } w = 0; \\ |w|\mathbb{Z} & \text{if } w \neq 0, \end{cases}$$

and vary continuously.

Now we are ready for our examples illustrating the hypotheses of Theorem 4.4.

(1) Let $G = \mathbb{R} \times \mathbb{R}$ act on $Y \times \mathbb{C}$ by $(s, r) \cdot (y, w) = (s \cdot y, r \cdot w)$. Then the stability subgroups are

$$S_{y,w} = \begin{cases} \{0\} \times \mathbb{R} & \text{if } w = 0; \\ \{0\} \times |w|\mathbb{Z} & \text{if } w \neq 0, \end{cases}$$

and vary continuously. Let $w_n \to 0$ in $\mathbb{C}$ and consider the sequence $x_n = (y_n, w_n) \to (\mathbf{0}, 0)$ in $Y \times \mathbb{C}$, where y_n are the orbit representatives for the action of $\mathbb{R}$ on Y described above. The stability subgroup at $(\mathbf{0}, 0)$ is the non-compact group $\{0\} \times \mathbb{R}$. We claim that the hypotheses of Theorem 4.4 hold with

$$t_n^{(1)} = (0, 0), \qquad t_n^{(2)} = (2n + \pi, 0), \quad \text{and} \quad W = [-1, 1] \times [-1, 1].$$

First, $t_n^{(1)} \cdot (y_n, w_n) = (y_n, w_n) \to (\mathbf{0}, 0)$ and $t_n^{(2)} \cdot (y_n, w_n) = ((2^{-2n-1}, 0, 0), 0) \to (\mathbf{0}, 0)$. Second,

$$W + t_n^{(1)} + S_{(y_n, w_n)} = [-1, 1] \times \{t + |w_n|\mathbb{Z} \colon t \in [-1, 1]\} \quad \text{and}$$
$$W + t_n^{(2)} + S_{(y_n, w_n)} = [-1 + 2n + \pi, 1 + 2n + \pi] \times \{t + |w_n|\mathbb{Z} \colon t \in [-1, 1]\},$$

and hence $(W + t_n^{(1)} + S_{(y_n, w_n)}) \cap (W + t_n^{(2)} + S_{(y_n, w_n)}) = \emptyset$ when $n \geqslant 1$. Third, $W S_{(\mathbf{0},0)} = [-1, 1] \times \mathbb{R}$ and $(\operatorname{Int} W) S_{(\mathbf{0},0)} = (-1, 1) \times \mathbb{R}$, and hence $W S_{(\mathbf{0},0)} \setminus ((\operatorname{Int} W) S_{(\mathbf{0},0)})$ has measure zero in $\mathbb{R} \times \mathbb{R}$. So the three hypotheses of Theorem 4.4 hold as claimed.

(2) Let $G = \mathbb{R} \times \mathbb{R}$ act on Y by $(s, r) \cdot y = s \cdot y$. Then the stability subgroup at y is $\{0\} \times \mathbb{R}$, and since the stability subgroups are constant they vary continuously. Consider $y_n \to \mathbf{0}$ in Y. Then, for the same reasons as in (1), the hypotheses of Theorem 4.4 hold with

$$t_n^{(1)} = (0, 0), \qquad t_n^{(2)} = (2n + \pi, 0), \quad \text{and} \quad W = [-1, 1] \times [-1, 1].$$

5. Measure accumulation and upper bounds on multiplicities

Throughout this section G is assumed to be abelian. Here we use bounds on measure accumulation to find upper bounds on $M_L(\pi, (\pi_n))$ where π_n and π are induced representations of $C_0(X) \rtimes G$. Theorem 5.2 has the same hypothesis as Proposition 5.1 but a stronger conclusion; in particular its proof uses that $M_L(\operatorname{Ind}(z, \tau), (\operatorname{Ind}(x_n, \tau_n)))$ is finite by Proposition 5.1.

Proposition 5.1. *Suppose that (G, X) is a second-countable transformation group with G abelian. Let $z \in X$ and let $(x_n)_{n \geqslant 1}$ be a sequence in X such that $S_{x_n} \to S_z$. Assume that $G \cdot z$ is locally closed in X and that S_z is compact. Let $M \in \mathbb{R}$ with $M \geqslant 1$, and suppose that there exists an open neighbourhood V of z in X such that $\phi_z^{-1}(V)$ is relatively compact and*

$$\nu_{x_n}\big(q_{x_n}\big(\phi_{x_n}^{-1}(V)\big)\big) \leqslant M \nu_z\big(q_z\big(\big(\phi_z^{-1}(V)\big)\big)\big) \tag{5.1}$$

frequently. Then $M_L(\operatorname{Ind}(z, \tau), (\operatorname{Ind}(x_n, \tau_n))) \leqslant \lfloor M^2 \rfloor$ for any $\tau, \tau_n \in \hat{G}$.

Proof. We may assume that $\operatorname{Ind}(x_n, \tau_n) \to \operatorname{Ind}(z, \tau)$, since $M_L(\operatorname{Ind}(z, \tau), (\operatorname{Ind}(x_n, \tau_n))) = 0$ otherwise. So by Proposition 4.3 it suffices to show that $M_L(\operatorname{Ind}(z, 1), (\operatorname{Ind}(x_n, 1))) \leqslant \lfloor M^2 \rfloor$. Since $M_L(\operatorname{Ind}(z, 1), (\operatorname{Ind}(x_n, 1))) \leqslant M_L(\operatorname{Ind}(z, 1), (\operatorname{Ind}(x_{n_i}, 1)))$ for any subsequence $(\operatorname{Ind}(x_{n_i}, 1))$, we may assume that (5.1) holds for all n.

Next, we claim that we may, by passing to a further subsequence, assume that $x_n \to z$. Note that $\ker \operatorname{Ind}(x_n, 1) \to \ker \operatorname{Ind}(z, 1)$ in $\operatorname{Prim}(C_0(X) \rtimes G)$. Since $\operatorname{Prim}(C_0(X) \rtimes G)$ is homeomorphic to $(X \times \hat{G})/\sim$ and since the quotient map $X \times \hat{G} \to (X \times \hat{G})/\sim$ is open [22, Theorem 5.3], there exists a subsequence $(x_{n_i}, 1)$ of $(x_n, 1)$ and $(y_i, \sigma_i) \in X \times \hat{G}$ such that $(x_{n_i}, 1) \sim (y_i, \sigma_i) \to (z, 1)$ in $X \times \hat{G}$. Thus $\overline{G \cdot y_i} = \overline{G \cdot x_{n_i}}$ and $\sigma_i|_{S_{x_{n_i}}} = 1$. Let (N_k) be a decreasing basic sequence of open neighbourhoods of z in X. There exists a subsequence (y_{i_k}) such that $y_{i_k} \in N_k$. Since $\overline{G \cdot y_{i_k}} = \overline{G \cdot x_{n_{i_k}}}$ there exists $g_k \in G$ such that $g_k \cdot x_{n_{i_k}} \in N_k$. Hence $g_k \cdot x_{n_{i_k}} \to z$. By [22, Corollary 4.8] $\operatorname{Ind}(x_{n_{i_k}}, 1)$ and $\operatorname{Ind}(g_k \cdot x_{n_{i_k}}, 1)$ are unitarily equivalent, and by the invariance of the measure we can replace $x_{n_{i_k}}$ with $g_k \cdot x_{n_{i_k}}$ in (5.1). So we may assume that $x_n \to z$ as claimed.

Now we will adapt the proof of [3, Theorem 3.1]. Fix $\epsilon > 0$ such that $M^2(1+\epsilon)^4 < \lfloor M^2 \rfloor + 1$. We will build a function $D \in C_c(G \times X)$ such that $\operatorname{Ind}(z, 1)(D^* * D)$ is a rank-one projection and

$$\operatorname{tr}\bigl(\operatorname{Ind}(x_n, 1)\bigl(D^* * D\bigr)\bigr) < M^2(1+\epsilon)^4 < \bigl\lfloor M^2 \bigr\rfloor + 1$$

eventually. (The function D is similar to the ones used in [16, Proposition 4.5], [22, Proposition 4.2] and [3, Theorem 3.1].) By the generalised lower semi-continuity result of [5, Theorem 4.3] we will have

$$\begin{aligned}\liminf_n \operatorname{tr}\bigl(\operatorname{Ind}(x_n, 1)\bigl(D^* * D\bigr)\bigr) &\geqslant M_L\bigl(\operatorname{Ind}(z, 1), \bigl(\operatorname{Ind}(x_n, 1)\bigr)\bigr) \operatorname{tr}\bigl(\operatorname{Ind}(z, 1)\bigl(D^* * D\bigr)\bigr) \\ &= M_L\bigl(\operatorname{Ind}(z, 1), \bigl(\operatorname{Ind}(x_n, 1)\bigr)\bigr),\end{aligned}$$

and the theorem will follow.

Let $\delta > 0$ such that

$$\delta < \frac{\epsilon \nu_z(q_z(\phi_z^{-1}(V)))}{1+\epsilon} < \nu_z\bigl(q_z\bigl(\phi_z^{-1}(V)\bigr)\bigr).$$

By the regularity of the measure ν_z there exists a compact subset W of G/S_z such that $W \subset q_z(\phi_z^{-1}(V))$ and

$$0 < \nu_z\bigl(q_z\bigl(\phi_z^{-1}(V)\bigr)\bigr) - \delta < \nu_z(W).$$

Since W is compact, there is a compact neighbourhood W_1 of W contained in the open set $q_z(\phi_z^{-1}(V))$ and a continuous function $g : G/S_z \to [0, 1]$ such that g is identically one on W and is identically zero off the interior of W_1. Then

$$\nu_z\bigl(q_z\bigl(\phi_z^{-1}(V)\bigr)\bigr) - \delta < \nu_z(W) \leqslant \int\limits_{G/S_z} g(\dot{u})^2 \, d\nu_z(\dot{u}) = \|g\|_{2,z}^2,$$

and hence

$$\frac{\nu_z(q_z(\phi_z^{-1}(V)))}{\|g\|_{2,z}^2} < 1 + \frac{\delta}{\|g\|_{2,z}^2} < 1 + \frac{\delta}{\nu_z(q_z(\phi_z^{-1}(V))) - \delta} < 1 + \epsilon. \tag{5.2}$$

Since $G\cdot z$ is locally closed in X it follows from [12, Theorem 1], applied to the locally compact Hausdorff transformation group $(G, G\cdot z)$, that $\phi_z : G \to G\cdot z$ induces a homeomorphism $\dot{s} \mapsto s\cdot z$ of G/S_z onto $G\cdot z$. So there is a continuous function $g_1 : W_1\cdot z \to [0,1]$ such that $g_1(\dot{u}\cdot z) = g(\dot{u})$ for $\dot{u} \in W_1$. Since $W_1\cdot z$ is a compact subset of the locally compact Hausdorff space X, it follows from Tietze's Extension Theorem (applied to the one-point compactification of X if necessary) that g_1 can be extended to a continuous function $g_2 : X \to [0,1]$. Because $W_1\cdot z$ is a compact subset of the open set V, there exists a compact neighbourhood P of $W_1\cdot z$ contained in V and a continuous function $h : X \to [0,1]$ such that h is identically one on $W_1\cdot z$ and is identically zero off the interior of P. Note that h has compact support contained in P. We set

$$f(x) = h(x)g_2(x).$$

Then $f \in C_c(X)$ with $0 \leqslant f \leqslant 1$ and $\operatorname{supp} f \subset \operatorname{supp} h \subset P \subset V$. Set $\tilde{f}_z(\dot{s}) = f(s\cdot z)$ so that $\tilde{f} : G/S_z \to X$. Note that

$$\|\tilde{f}_z\|_{2,z}^2 = \int\limits_{G/S_z} \tilde{f}(\dot{u}\cdot z)^2\,d\nu_z(\dot{u}) = \int\limits_{G/S_z} h(\dot{u}\cdot z)^2 g_2(\dot{u}\cdot z)^2\,d\nu_z(\dot{u}) \geqslant \int\limits_{W_1} g(\dot{u})^2\,d\nu_z(\dot{u}) = \|g\|_{2,z}^2 \tag{5.3}$$

since h is identically one on $W_1\cdot z$ and the support of g is contained in W_1. We now set

$$F(x) = \frac{f(x)}{\|\tilde{f}_z\|_{2,z}}.$$

Now $F \in C_c(X)$ and $F_x(s) = F(s\cdot x) \neq 0$ implies $s \in \phi_x^{-1}(V)$ by our choice of h. Since $\phi_z^{-1}(V)$ is relatively compact, $\operatorname{supp} F_z$ is compact. Write $\tilde{F}_z(\dot{s}) = F(s\cdot z)$ and note that $\|\tilde{F}_z\|_{2,z} = 1$.

Recall that S_z is compact by assumption. Choose $b \in C_c(G\times X)$ such that $0 \leqslant b \leqslant 1/\alpha_z(S_z)$ and b is identically $1/\alpha_z(S_z)$ on the set $(\operatorname{supp} F_z)S_z(\operatorname{supp} F_z)^{-1} \times \operatorname{supp} F$; we may obtain that $\operatorname{supp} b \subset N \times X$ where $N = N^{-1}$ is a compact subset of G containing S_z. Set

$$B(r,x) = F(x)F\big(r^{-1}\cdot x\big)b\big(r^{-1},x\big) \quad \text{and} \quad D = \frac{1}{2}\big(B + B^*\big).$$

We have

$$\begin{aligned}
\big(\operatorname{Ind}(x,1)(B)\xi\big)(\dot{s}) &= \int\limits_G B(r, s\cdot x)\xi\big(r^{-1}\dot{s}\big)\,d\mu(r) \\
&= \int\limits_G F(s\cdot x)F\big(r^{-1}s\cdot x\big)b\big(r^{-1}, s\cdot x\big)\xi\big(r^{-1}\dot{s}\big)\,d\mu(r) \\
&= F(s\cdot x)\int\limits_G F(u\cdot x)b\big(us^{-1}, s\cdot x\big)\xi(\dot{u})\,d\mu(u) \\
&= F(s\cdot x)\int\limits_{G/S_x} F(u\cdot x)\int\limits_{S_x} b\big(uts^{-1}, s\cdot x\big)\,d\alpha_x(t)\xi(\dot{u})\,d\nu_x(\dot{u})
\end{aligned}$$

so that

$$\begin{aligned}&\big(\mathrm{Ind}(x,1)(D)\xi\big)(\dot s)\\&\quad=\frac{1}{2}F(s\cdot x)\int_{G/S_x}F(u\cdot x)\bigg(\int_{S_x}\big(b\big(us^{-1}t,s\cdot x\big)+b\big(su^{-1}t,u\cdot x\big)\big)\,d\alpha_x(t)\bigg)\xi(\dot u)\,d\nu_x(\dot u)\end{aligned}$$

since G is abelian.

If $F(s\cdot z)$ and $F(u\cdot z)$ are nonzero then $s,u\in\operatorname{supp}F_z$, and hence $b(uts^{-1},s\cdot z)+b(stu^{-1},u\cdot z)=2/\alpha_z(S_z)$ for all $t\in S_z$. It follows that

$$\big(\mathrm{Ind}(z,1)(D)\xi\big)(\dot s)=F(s\cdot z)\int_{G/S_z}F(u\cdot z)\xi(\dot u)\,d\nu_z(\dot u)=(\xi,\tilde F_z)\tilde F_z(\dot s).$$

Thus $\mathrm{Ind}(z,1)(D)$, and hence $\mathrm{Ind}(z,1)(D^*D)$, is the rank-one projection determined by the unit vector $\tilde F_z\in L^2(G/S_z,\nu_z)$.

Recall that we are assuming that (5.1) holds for all n and set $E_n=\{s\in G:\ F(s\cdot x_n)\neq 0\}$. Then each $q_{x_n}(E_n)$ is open, hence measurable, with

$$\nu_{x_n}\big(q_{x_n}(E_n)\big)\leqslant\nu_{x_n}\big(q_{x_n}\big(\phi_{x_n}^{-1}(V)\big)\big)\leqslant M\nu_z\big(q_z\big(\phi_z^{-1}(V)\big)\big)<\infty. \tag{5.4}$$

Note that $\mathrm{Ind}(x_n,1)(D)$ is a kernel operator with kernel

$$K_n(\dot s,\dot u):=\frac{1}{2}F(s\cdot x_n)F(u\cdot x_n)\bigg(\int_{S_{x_n}}\big(b\big(us^{-1}t,s\cdot x_n\big)+b\big(su^{-1}t,u\cdot x_n\big)\big)\,d\alpha_{x_n}(t)\bigg).$$

To see that $\mathrm{Ind}(x_n,1)(D)$ is a Hilbert–Schmidt operator on $L^2(G/S_{x_n})$, we need to see that K_n is in $L^2(G/S_{x_n}\times G/S_{x_n})$. The support of K_n is contained in $q_{x_n}(E_n)\times q_{x_n}(E_n)$, which has finite measure by (5.4). Note that K_n is continuous, hence measurable, since F is continuous and $b\in C_c(G\times X)$. To see that K_n is bounded, set

$$\Upsilon_n(s,u):=\int_{S_{x_n}}\big(b\big(us^{-1}t,s\cdot x_n\big)+b\big(su^{-1}t,u\cdot x_n\big)\big)\,d\alpha_{x_n}(t),$$

and note that Υ_n is constant on $S_{x_n}\times S_{x_n}$-cosets. Recall that $0\leqslant b\leqslant 1/\alpha_z(S_z)$ and that $\operatorname{supp}b\subset N\times X$ where $N=N^{-1}$ and $S_z\subset N$. If $us^{-1}t\notin N$ for all $t\in S_{x_n}$ then $\Upsilon(s,u)=0$. If $us^{-1}t_0\in N$ for some $t_0\in S_{x_n}$ then we may assume that $us^{-1}\in N$ (because $\Upsilon_n(s,u)=\Upsilon_n(t_0^{-1}s,u)=\Upsilon_n(t_0^{-1}s,ut_0^{-1})$). Thus

$$\Upsilon_n(s,u)\leqslant\frac{2}{\alpha_z(S_z)}\alpha_{x_n}\big(\big\{t\in S_{x_n}:\ us^{-1}t\in N\big\}\big)\leqslant\frac{2}{\alpha_z(S_z)}\alpha_{x_n}\big(S_{x_n}\cap N^2\big).$$

Let $\eta\in C_c(G)^+$ such that η is identically one on N^2. It follows from our choice of continuous Haar measures on the closed subgroups of G that $H\mapsto\int_H\eta(t)\,d\alpha_H(t)$ is a continuous function on Σ. Since $S_{x_n}\to S_z$ by assumption, there exists n_0 such that, for $n\geqslant n_0$,

$$\Upsilon_n(s,u) \leqslant \frac{2}{\alpha_z(S_z)} \int\limits_{S_{x_n}} \eta(t)\,d\alpha_{x_n}(t) \leqslant \frac{2}{\alpha_z(S_z)} \int\limits_{S_z} \eta(t)\,d\alpha_z(t)(1+\epsilon) = 2(1+\epsilon). \tag{5.5}$$

Hence $0 \leqslant K_n(\dot{s},\dot{u}) \leqslant \|F\|_\infty^2(1+\epsilon)$ when $n \geqslant n_0$.

Let $n \geqslant n_0$. Then $\mathrm{Ind}(x_n,1)(D)$ is the self-adjoint Hilbert–Schmidt operator with kernel K_n. It follows that $\mathrm{Ind}(x_n,1)(D^* * D)$ is a trace-class operator with

$$\mathrm{tr}\big(\mathrm{Ind}(x_n,1)\big(D^* * D\big)\big) = \|K_n\|_{2,x_n}^2$$

(see, for example, [19, Proposition 3.4.16]). To estimate the trace we note, using (5.4) and (5.3), that

$$\int\limits_{G/S_{x_n}} F(s\cdot x_n)^2\,d\nu_{x_n}(\dot{s}) \leqslant \frac{\nu_{x_n}(q_{x_n}(E_n))}{\|f_z\|_{2,z}^2} \leqslant \frac{M\nu_z(q_z(\phi_z^{-1}(V)))}{\|g\|_{2,z}^2}. \tag{5.6}$$

An application of Fubini's Theorem gives

$$\begin{aligned}
&\mathrm{tr}\big(\mathrm{Ind}(x_n,1)\big(D^* * D\big)\big) \\
&\quad = \frac{1}{4} \int\limits_{G/S_{x_n}} \int\limits_{G/S_{x_n}} F(s\cdot x_n)^2 F(u\cdot x_n)^2 \Upsilon_n(s,u)^2\,d\nu_{x_n}(\dot{s})\,d\nu_{x_n}(\dot{u}) &(5.7)\\
&\quad \leqslant \int\limits_{G/S_{x_n}} \int\limits_{G/S_{x_n}} F(s\cdot x_n)^2 F(u\cdot x_n)^2 (1+\epsilon)^2\,d\nu_{x_n}(\dot{s})\,d\nu_{x_n}(\dot{u}) \quad \big(\text{using (5.5)}\big)\\
&\quad = (1+\epsilon)^2\bigg(\int\limits_{G} F(s\cdot x_n)^2\,d\nu_{x_n}(\dot{s})\bigg)^2\\
&\quad \leqslant \frac{(1+\epsilon)^2 M^2 \nu_z(q_z(\phi_z^{-1}(V)))^2}{\|g\|_{2,z}^4} \quad \big(\text{using (5.6)}\big)\\
&\quad < M^2(1+\epsilon)^4 \quad \big(\text{using (5.2)}\big).
\end{aligned}$$

Finally,

$$M_L\big(\mathrm{Ind}(z,1),\big(\mathrm{Ind}(x_n,1)\big)\big) \leqslant \liminf_n \mathrm{tr}\big(\mathrm{Ind}(x_n,1)\big(D^* * D\big)\big) \leqslant M^2(1+\epsilon)^4 < \lfloor M^2\rfloor + 1,$$

and hence $M_L(\mathrm{Ind}(z,1),(\mathrm{Ind}(x_n,1))) \leqslant \lfloor M^2\rfloor$. $\square$

Theorem 5.2. *Suppose that (G,X) is a second-countable transformation group with G abelian. Let $z \in X$ and let $(x_n)_{n\geqslant 1}$ be a sequence in X such that $S_{x_n} \to S_z$. Assume that $G\cdot z$ is locally closed in X and that S_z is compact. Let $M \in \mathbb{R}$ with $M \geqslant 1$, and suppose that there exists an open neighbourhood V of z in X such that $\phi_z^{-1}(V)$ is relatively compact and*

$$\nu_{x_n}\big(q_{x_n}\big(\phi_{x_n}^{-1}(V)\big)\big) \leqslant M\nu_z\big(q_z\big(\big(\phi_z^{-1}(V)\big)\big)\big) \tag{5.8}$$

frequently. Then $M_L(\mathrm{Ind}(z,\tau),(\mathrm{Ind}(x_n,\tau_n))) \leqslant \lfloor M\rfloor$ for any $\tau,\tau_n \in \hat{G}$.

Proof. As in the proof of Theorem 5.1, it suffices to prove $M_L(\mathrm{Ind}(z,1),(\mathrm{Ind}(x_n,1))) \leqslant \lfloor M \rfloor$ where $x_n \to z$ and (5.8) holds for all n. Next, we claim that we may as well assume that $G \cdot z$ is the unique limit of $(G \cdot x_n)$ in X/G.

Since $M_L(\mathrm{Ind}(z,1),(\mathrm{Ind}(x_n,1))) \leqslant \lfloor M^2 \rfloor < \infty$ by Theorem 5.1, $\{\mathrm{Ind}(z,1)\}$ is open in the set of limits of $(\mathrm{Ind}(x_n,1))_n$ [3, Proposition 3.4]. So there is an open neighbourhood U of $\mathrm{Ind}(z,1)$ in the spectrum of $C_0(X) \rtimes G$ such that $\mathrm{Ind}(z,1)$ is the unique limit of $(\mathrm{Ind}(x_n,1))_n$ in U. The map $\mathrm{Ind} : X \to (C_0(X) \rtimes G)^\wedge$, $x \mapsto \mathrm{Ind}(x,1)$ is continuous and factors through the T_0-isation of X/G [22, Lemma 4.10]. Hence $Y := \mathrm{Ind}^{-1}(U)$ is an open G-invariant neighbourhood of z in X. Note that $x_n \in Y$ eventually, and that $x \in Y$ implies $\phi_x^{-1}(V) = \phi_x^{-1}(V \cap Y)$. Now we argue as in [3, Proof of Theorem 3.5] that if $G \cdot x_n \to G \cdot y$ for some $y \in Y$, then $G \cdot y = G \cdot z$ since $G \cdot z$ is locally closed. Since $C_0(Y) \rtimes G$ is an ideal in $C_0(X) \rtimes G$, we may compute multiplicities in the ideal instead [6, Proposition 5.3], so we may replace X by Y and assume that $G \cdot z$ is the unique limit of $G \cdot x_n$ in X/G, as claimed.

Fix $\epsilon > 0$ such that $M(1+\epsilon)^4 < \lfloor M \rfloor + 1$ and choose $\gamma > 0$ such that

$$\gamma < \frac{\epsilon \nu_z(q_z(\phi_z^{-1}(V)))}{1+\epsilon} < \nu_z\bigl(q_z\bigl(\phi_z^{-1}(V)\bigr)\bigr). \tag{5.9}$$

It follows from the regularity of the measure ν_z, as in [3, Proof of Lemma 3.3], that there exists an open relatively compact neighbourhood V_1 of z such that $\overline{V_1} \subset V$ and

$$\begin{aligned} 0 < \nu_z\bigl(q_z\bigl(\phi_z^{-1}(V)\bigr)\bigr) - \gamma &< \nu_z\bigl(q_z\bigl(\phi_z^{-1}(V_1)\bigr)\bigr) \leqslant \nu_z\bigl(q_z\bigl(\phi_z^{-1}(\overline{V_1})\bigr)\bigr) \\ &\leqslant \nu_z\bigl(q_z\bigl(\phi_z^{-1}(V)\bigr)\bigr) < \nu_z\bigl(q_z\bigl(\phi_z^{-1}(V_1)\bigr)\bigr) + \gamma. \end{aligned}$$

(The reason for passing from V to V_1 is that we will later apply [3, Lemma 3.2] to the compact neighbourhood $\overline{V_1}$ and, in contrast to what could happen with $\overline{V}$, we can control $\nu_z(q_z(\phi_z^{-1}(\overline{V_1})))$ relative to $\nu_z(q_z(\phi_z^{-1}(V_1)))$.)

Recall that we are assuming that (5.8) holds for all n. Thus

$$\begin{aligned} \nu_{x_n}\bigl(q_{x_n}\bigl(\phi_{x_n}^{-1}(V_1)\bigr)\bigr) &\leqslant \nu_{x_n}\bigl(q_{x_n}\bigl(\phi_{x_n}^{-1}(V)\bigr)\bigr) \\ &\leqslant M\nu_z\bigl(q_z\bigl(\phi_z^{-1}(V)\bigr)\bigr) \quad \bigl(\text{by (5.8)}\bigr) \\ &< M\bigl(\nu_z\bigl(q_z\bigl(\phi_z^{-1}(V_1)\bigr)\bigr) + \gamma\bigr) \\ &< M\nu_z\bigl(q_z\bigl(\phi_z^{-1}(V_1)\bigr)\bigr) + M\epsilon\bigl(\nu_z\bigl(q_z\bigl(\phi_z^{-1}(V)\bigr)\bigr) - \gamma\bigr) \quad \bigl(\text{by (5.9)}\bigr) \\ &< M\nu_z\bigl(q_z\bigl(\phi_z^{-1}(V_1)\bigr)\bigr) + M\epsilon\nu_z\bigl(q_z\bigl(\phi_z^{-1}(V_1)\bigr)\bigr) \\ &= M(1+\epsilon)\nu_z\bigl(q_z\bigl(\phi_z^{-1}(V_1)\bigr)\bigr) \end{aligned} \tag{5.10}$$

for all n. Since

$$\frac{\nu_z(q_z(\phi_z^{-1}(V_1)))(\nu_z(q_z(\phi_z^{-1}(V_1))) + \gamma + \frac{1}{j})}{(\nu_z(q_z(\phi_z^{-1}(V_1))) - \frac{1}{j})^2} \to 1 + \frac{\gamma}{\nu_z(q_z(\phi_z^{-1}(V_1)))} < 1 + \epsilon$$

as $j \to \infty$, there exists $\delta > 0$ such that $\delta < \nu_z(q_z(\phi_z^{-1}(V_1)))$ and

$$\frac{\nu_z(q_z(\phi_z^{-1}(V_1)))(\nu_z(q_z(\phi_z^{-1}(\overline{V_1}))) + \delta)}{(\nu_z(q_z(\phi_z^{-1}(V_1))) - \delta)^2} < \frac{\nu_z(q_z(\phi_z^{-1}(V_1)))(\nu(\phi_z^{-1}(V_1)) + \gamma + \delta)}{(\nu_z(q_z(\phi_z^{-1}(V_1))) - \delta)^2}$$
$$< 1 + \epsilon. \tag{5.11}$$

Next we construct a function $F \in C_c(X)$ with support contained in V_1. By the regularity of the measure ν_z there exists a compact subset W of the open set $q_z(\phi_z^{-1}(V_1))$ such that $0 < \nu_z(q_z(\phi_z^{-1}(V_1))) - \delta < \nu_z(W)$. Since W is compact, there is a compact neighbourhood W_1 of W contained in $q_z(\phi_z^{-1}(V_1))$ and a continuous function $g : G/S_z \to [0, 1]$ such that g is identically one on W and is identically zero off the interior of W_1. Then

$$\nu_z\big(q_z\big(\phi_z^{-1}(V_1)\big)\big) - \delta < \nu_z(W) \leqslant \int\limits_{G/S_z} g(\dot{t})^2\, d\nu_z(\dot{t}) = \|g\|_{2,z}^2. \tag{5.12}$$

We now construct g_1, g_2, P, h, f and F as in the proof of Proposition 5.1, but we note that this time $\operatorname{supp} f \subset \operatorname{supp} h \subset P \subset V_1$. So if $F_x(s) = F(s \cdot x) \neq 0$ then $s \in \phi_x^{-1}(V_1)$. As before

$$\|\tilde{f}_z\|_{2,z}^2 \geqslant \|g\|_{2,z}^2 \tag{5.13}$$

and $\|\tilde{F}_z\|_{2,z} = 1$.

Let K be an open relatively compact symmetric neighbourhood of $(\operatorname{supp} F_z)S_z(\operatorname{supp} F_z)^{-1}$ in G and L an open relatively compact neighbourhood of $\operatorname{supp} F$ in X. Choose $b \in C_c(G \times X)$ such that $0 \leqslant b \leqslant 1/\alpha_z(S_z)$, b is identically $1/\alpha_z(S_z)$ on the set $(\operatorname{supp} F_z)S_z(\operatorname{supp} F_z)^{-1} \times \operatorname{supp} F$ and b is identically zero off $K \times L$. (Thus b is as in Theorem 5.1, but we have rounded it off with an open set.) Set

$$B(r, x) = F(x)F\big(r^{-1} \cdot x\big)b\big(r^{-1}, x\big) \quad \text{and} \quad D = \frac{1}{2}\big(B + B^*\big).$$

Again, $\operatorname{Ind}(z, 1)(D)$, and hence $\operatorname{Ind}(z, 1)(D^* * D)$, is the rank-one projection determined by the unit vector $\tilde{F}_z \in L^2(G/S_z, \nu)$. From (5.7) there exists n_0 such that

$$\operatorname{tr}\big(\operatorname{Ind}(x_n, 1)\big(D^* * D\big)\big) = \frac{1}{4} \int\limits_{G/S_{x_n}} F(s \cdot x_n)^2 \Bigg(\int\limits_{G/S_{x_n}} F(u \cdot x_n)^2 \Upsilon_n(s, u)^2 d\nu_z(\dot{u}) \Bigg) d\nu_z \nu(\dot{s})$$

where

$$\Upsilon_n(s, u) := \int\limits_{S_{x_n}} \big(b\big(us^{-1}t, s \cdot x_n\big) + b\big(su^{-1}t, u \cdot x_n\big)\big)\, d\alpha_{x_n}(t) \leqslant 2(1 + \epsilon)$$

when $n \geqslant n_0$. For fixed $s \in G$, $F(u \cdot x_n)\Upsilon_n(s, u) \neq 0$ implies that $u \in \phi_{x_n}^{-1}(V_1)$ and $us^{-1}t = uts^{-1} \in K$ for some $t \in S_{x_n}$ because b is identically zero off $K \times L$. So $u \in \phi_{x_n}^{-1}(V_1) \cap KsS_{x_n}$. Thus if $n \geqslant n_0$,

$$
\begin{aligned}
&\operatorname{tr}\bigl(\operatorname{Ind}(x_n,1)\bigl(D^* * D\bigr)\bigr)\\
&\quad\leqslant (1+\epsilon)^2 \int\limits_{s\in q_{x_n}(\phi_{x_n}^{-1}(V_1))} F(s\cdot x_n)^2 \Biggl(\int\limits_{u\in q_{x_n}(\phi_{x_n}^{-1}(V_1)\cap KsS_{x_n})} F(u\cdot x_n)^2\, d\nu_{x_n}(\dot u)\Biggr)\, d\nu_{x_n}(\dot s)\\
&\quad\leqslant \frac{(1+\epsilon)^2}{\|\tilde f_z\|_{2,z}^4} \int\limits_{s\in q_{x_n}(\phi_{x_n}^{-1}(V_1))} 1 \Biggl(\int\limits_{u\in q_{x_n}(\phi_{x_n}^{-1}(V_1)\cap KsS_{x_n})} 1\, d\nu_{x_n}(\dot u)\Biggr)\, d\nu_{x_n}(\dot s).
\end{aligned}
$$

By the regularity of ν_z there exists an open neighbourhood U of $q_z(\phi_z^{-1}(\overline{V_1}))$ such that $\nu_z(U) < \nu_z(q_z(\phi_z^{-1}(\overline{V_1}))) + \delta/2$. Since $q_z(\phi_z^{-1}(\overline{V_1}))$ is compact, it has an open relatively compact neighbourhood A such that $\overline{A} \subset U$. By [3, Lemma 3.2], applied with $\overline{V_1}$, $\overline{K}$ and $q_z^{-1}(A)$, there exists $n_1 > n_0$ such that, for every $n \geqslant n_1$ and every $s \in \phi_{x_n}^{-1}(\overline{V_1})$ there exists $r \in \phi_z^{-1}(\overline{V_1})$ with $\overline{K}s \cap \phi_{x_n}^{-1}(\overline{V_1}) \subset q_z^{-1}(A)r^{-1}s$. Hence $\overline{K}sS_{x_n} \cap \phi_{x_n}^{-1}(\overline{V_1}) \subset q_z^{-1}(A)r^{-1}sS_{x_n}$. Since ν_{x_n} is right-invariant we have

$$
\begin{aligned}
\nu_{x_n}\bigl(q_{x_n}\bigl(\overline{K}sS_{x_n} \cap \phi_{x_{n_i}}^{-1}(\overline{V_1})\bigr)\bigr) &\leqslant \nu_{x_n}\bigl(q_{x_n}\bigl(q_z^{-1}(A)S_{x_n}\bigr)\bigr)\\
&= \nu_{x_n}\bigl(q_{x_n}\bigl(q_z^{-1}(A)\bigr)\bigr) \leqslant \nu_{x_n}\bigl(q_{x_n}\bigl(q_z^{-1}(\overline{A})\bigr)\bigr).
\end{aligned}
$$

Since $\overline{A}$ and S_z are both compact, $q_z^{-1}(\overline{A})$ is compact, and by Lemma 3.4 there exists $n_2 > n_1$ such that $n \geqslant n_2$ implies that

$$
\nu_{x_n}\bigl(q_{x_n}\bigl(q_z^{-1}(\overline{A})\bigr)\bigr) \leqslant \nu_z\bigl(q_z\bigl(q_z^{-1}(\overline{A})\bigr)\bigr) + \delta/2 < \nu_z\bigl(q_z\bigl(\phi_z^{-1}(\overline{V_1})\bigr)\bigr) + \delta.
$$

So, provided $n \geqslant n_2$,

$$
\begin{aligned}
\operatorname{tr}\bigl(\operatorname{Ind}(x_n,1)\bigl(D^* * D\bigr)\bigr) &\leqslant \frac{(1+\epsilon)^2 \nu_{x_n}(q_{x_n}(\phi_{x_n}^{-1}(V_1)))(\nu_z(q_z(\phi_z^{-1}(\overline{V_1}))) + \delta)}{\|\tilde f_z\|_{2,z}^4}\\
&< \frac{M(1+\epsilon)^3 \nu_z(q_z(\phi_z^{-1}(V_1)))(\nu_z(q_z(\phi_z^{-1}(\overline{V_1}))) + \delta)}{\|\tilde f_z\|_{2,z}^4} \quad \text{using (5.10)}\\
&< \frac{M(1+\epsilon)^3 \nu_z(q_z(\phi_z^{-1}(V_1)))(\nu_z(q_z(\phi_z^{-1}(\overline{V_1}))) + \delta)}{\|g\|_{2,z}^4} \quad \text{using (5.13)}\\
&\leqslant \frac{M(1+\epsilon)^3 \nu_z(q_z(\phi_z^{-1}(V_1)))(\nu_z(q_z(\phi_z^{-1}(\overline{V_1}))) + \delta)}{(\nu_z(q_z(\phi_z^{-1}(V_1))) - \delta)^2} \quad \text{using (5.12)}\\
&< M(1+\epsilon)^4 \quad \text{using (5.11).}
\end{aligned}
$$

By generalised lower semi-continuity [5, Theorem 4.3],

$$
\begin{aligned}
\liminf_n \operatorname{tr}\bigl(\operatorname{Ind}(x_n,1)\bigl(D^* * D\bigr)\bigr) &\geqslant M_L\bigl(\operatorname{Ind}(z,1), \bigl(\operatorname{Ind}(x_n,1)\bigr)\bigr) \operatorname{tr}\bigl(\operatorname{Ind}(z,1)\bigl(D^* * D\bigr)\bigr)\\
&= M_L\bigl(\operatorname{Ind}(z,1), \bigl(\operatorname{Ind}(x_n,1)\bigr)\bigr).
\end{aligned}
$$

We now have

$$M_L\big(\mathrm{Ind}(z,1),\big(\mathrm{Ind}(x_n,1)\big)\big) \leqslant \liminf_n \mathrm{tr}\big(\mathrm{Ind}(x_n,1)\big(D^* * D\big)\big) \leqslant M(1+\epsilon)^4 < \lfloor M\rfloor + 1,$$

and hence $M_L(\mathrm{Ind}(z,1),(\mathrm{Ind}(x_n,1))) \leqslant \lfloor M\rfloor$. □

6. The main theorem

In this section we combine the results from Sections 4–5 to obtain our main theorem.

Theorem 6.1. *Suppose that (G,X) is a second-countable transformation group with G abelian. Let $z \in X$ and let $(x_n)_{n\geqslant 1}$ be a sequence in X such that $S_{x_n} \to S_z$ in Σ. Assume that $G\cdot z$ is locally closed in X and that S_z is compact. Let $k \in \mathbb{P}$. Then the following are equivalent*:

(1) *the sequence $(x_n)_n$ converges k-times in X/G to z*;
(2) $M_L(\mathrm{Ind}(z,1),(\mathrm{Ind}(x_n,1))) \geqslant k$;
(3) *there exist $\tau_n, \tau \in \hat{G}$ such that* $M_L(\mathrm{Ind}(z,\tau),(\mathrm{Ind}(x_n,\tau_n))) \geqslant k$;
(4) *there exist $\tau_n, \tau \in \hat{G}$ such that* $(\mathrm{Ind}(x_n,\tau_n)) \to \mathrm{Ind}(z,\tau)$, *and whenever $\sigma_n, \sigma \in \hat{G}$ such that* $(\mathrm{Ind}(x_n,\sigma_n)) \to \mathrm{Ind}(z,\sigma)$, $M_L(\mathrm{Ind}(z,\sigma),(\mathrm{Ind}(x_n,\sigma_n))) \geqslant k$;
(5) *for every open neighbourhood V of z such that $\phi_z^{-1}(V)$ is relatively compact we have*

$$\liminf_n \nu_{x_n}\big(q_{x_n}\big(\phi_{x_n}^{-1}(V)\big)\big) \geqslant k\nu_z\big(q_z\big(\phi_x^{-1}(V)\big)\big);$$

(6) *there exists a real number $R > k-1$ such that for every open neighbourhood V of z with $\phi_z^{-1}(V)$ relatively compact we have*

$$\liminf_n \nu_{x_n}\big(q_{x_n}\big(\phi_{x_n}^{-1}(V)\big)\big) \geqslant R\nu_z\big(q_z\big(\phi_x^{-1}(V)\big)\big);$$

(7) *there exists a decreasing basic sequence of compact neighbourhoods $(W_m)_{m\geqslant 1}$ of z such that, for each $m \geqslant 1$,*

$$\liminf_n \nu_{x_n}\big(q_{x_n}\big(\phi_{x_n}^{-1}(W_m)\big)\big) > (k-1)\nu_z\big(q_z\big(\phi_x^{-1}(W_m)\big)\big).$$

We show that (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (2) ⇒ (5) ⇒ (6) ⇒ (7) ⇒ (1). The reason for going from (4) to (5) via (2) is that (2)–(4) are very similar and (2) is the least complicated to work with.

Proof. (1) ⇒ (2). Assume the sequence $(x_n)_n$ converges k-times in X/G to z. Then $G\cdot x_n \to G\cdot z$, and hence $\mathrm{Ind}(x_n,1) \to \mathrm{Ind}(z,1)$. Now $M_L(\mathrm{Ind}(z,1),(\mathrm{Ind}(x_n,1))) \geqslant k$ by Corollary 4.6.

(2) ⇒ (3). Take $\tau_n = \tau = 1$.

(3) ⇒ (4). Assume (3). Since $M_L(\mathrm{Ind}(z,\tau),(\mathrm{Ind}(x_n,\tau_n))) > 0$, $(\mathrm{Ind}(x_n,\tau_n)) \to \mathrm{Ind}(z,\tau)$. Suppose $\sigma_n, \sigma \in \hat{G}$ such that $(\mathrm{Ind}(x_n,\sigma_n)) \to \mathrm{Ind}(z,\sigma)$. By two applications of Proposition 4.3 we have

$$\begin{aligned} M_L\big(\mathrm{Ind}(z,\sigma),\big(\mathrm{Ind}(x_n,\sigma_n)\big)\big) &= M_L\big(\mathrm{Ind}(z,1),\big(\mathrm{Ind}(x_n,1)\big)\big) \\ &= M_L\big(\mathrm{Ind}(z,\tau),\big(\mathrm{Ind}(x_n,\tau_n)\big)\big) \geqslant k, \end{aligned}$$

so (4) holds.

(4) $\Rightarrow$ (2). Assume (4). Let $\tau, \tau_n \in \hat{G}$ such that $(\mathrm{Ind}(x_n, \tau_n)) \to \mathrm{Ind}(z, \tau)$ and $M_L(\mathrm{Ind}(z, \tau), (\mathrm{Ind}(x_n, \tau_n))) \geqslant k$. Then $M_L(\mathrm{Ind}(z, 1), (\mathrm{Ind}(x_n, 1))) \geqslant k$ by Proposition 4.3, giving (2).

(2) $\Rightarrow$ (5). Assume (2), that is, $M_L(\mathrm{Ind}(z, 1), (\mathrm{Ind}(x_n, 1))) \geqslant k$. Let $\epsilon > 0$. Then $M_L(\mathrm{Ind}(z, 1), (\mathrm{Ind}(x_n, 1))) > \lfloor k - \epsilon \rfloor$. By Theorem 5.2, for every open neighbourhood V of z such that $\phi_z^{-1}(V)$ is relatively compact,

$$\nu_{x_n}\big(q_{x_n}\big(\phi_{x_n}^{-1}(V)\big)\big) > (k - \epsilon)\nu_z\big(q_z\big(\phi_z^{-1}(V)\big)\big)$$

eventually. Thus (5) holds.

(5) $\Rightarrow$ (6). Take $R = k$.

For (6) $\Rightarrow$ (7) we need the following lemma concerning accumulation of measure.

Lemma 6.2. *Suppose that (G, X) is a transformation group. Let $z \in X$ and $(x_n)_{n \geqslant 1}$ be a sequence in X. Assume that $G \cdot z$ is locally closed in X and that S_z is compact. Let $k \in \mathbb{P}$, and assume that there exists a real number $R > k - 1$ such that for every open neighbourhood U of z with $\phi_z^{-1}(U)$ relatively compact we have*

$$\liminf_n \nu_{x_n}\big(q_{x_n}\big(\phi_{x_n}^{-1}(U)\big)\big) \geqslant R\nu_z\big(q_z\big(\phi_z^{-1}(U)\big)\big).$$

Then given an open neighbourhood V of z such that $\phi_z^{-1}(V)$ is relatively compact, there exists a compact neighbourhood N of z with $N \subset V$ such that

$$\liminf_n \nu_{x_n}\big(q_{x_n}\big(\phi_{x_n}^{-1}(N)\big)\big) > (k - 1)\nu_z\big(q_z\big(\phi_z^{-1}(N)\big)\big).$$

Proof. Fix $0 < \gamma < (\frac{R-k+1}{R})\nu_z(q_z(\phi_z^{-1}(V)))$. By the regularity of ν_z, as in [3, Proof of Lemma 3.3], there exists an open relatively compact neighbourhood V_1 of z with $\overline{V_1} \subset V$ and $\nu_z(q_z(\phi_z^{-1}(V))) - \gamma < \nu_z(q_z(\phi_z^{-1}(V_1)))$. Since $\phi_z^{-1}(V_1)$ is relatively compact we have

$$\begin{aligned}
\liminf_n \nu_z\big(q_{x_n}\big(\phi_{x_n}^{-1}(\overline{V}_1)\big)\big) &\geqslant \liminf_n \nu_{x_n}\big(q_{x_n}\big(\phi_{x_n}^{-1}(V_1)\big)\big) \\
&\geqslant R\nu_z\big(q_z\big(\phi_z^{-1}(V_1)\big)\big) \quad \text{by hypothesis} \\
&> R\big(\nu_z\big(q_z\big(\phi_z^{-1}(V)\big)\big) - \gamma\big) \\
&> (k-1)\nu_z\big(q_z\big(\phi_z^{-1}(V)\big)\big) \quad \text{by our choice of } \gamma \\
&\geqslant (k-1)\nu_z\big(q_z\big(\phi_z^{-1}(\overline{V}_1)\big)\big).
\end{aligned}$$

So we may take $N = \overline{V}_1$. $\square$

We now continue with the proof of Theorem 6.1.

(6) $\Rightarrow$ (7). Assume (6). Let $(V_j)_{j \geqslant 1}$ be a decreasing basic sequence of open neighbourhoods of z such that $\phi_z^{-1}(V_1)$ is relatively compact (such neighbourhoods exist by [3, Lemma 2.1]). By Lemma 6.2 there exists a compact neighbourhood W_1 of z such that $W_1 \subset V_1$ and

$$\liminf_n \nu_{x_n}\big(q_{x_n}\big(\phi_{x_n}^{-1}(W_1)\big)\big) > (k - 1)\nu_z\big(q_z\big(\phi_z^{-1}(W_1)\big)\big).$$

Now assume there are compact neighbourhoods $W_1, W_2, \dots, W_m$ of z with $W_1 \supset W_2 \supset \cdots \supset W_m$ such that

$$W_i \subset V_i \quad \text{and} \quad \liminf_n \nu_{x_n}\big(q_{x_n}\big(\phi_{x_n}^{-1}(W_i)\big)\big) > (k-1)\nu_z\big(q_z\big(\phi_z^{-1}(W_i)\big)\big) \tag{6.1}$$

for $1 \leqslant i \leqslant m$. Apply Lemma 6.2 to $(\operatorname{Int} W_m) \cap V_{m+1}$ to obtain a compact neighbourhood W_{m+1} of z such that $W_{m+1} \subset (\operatorname{Int} W_m) \cap V_{m+1}$ and (6.1) holds for $i = m+1$. This gives (7).

(7) ⇒ (1). Assume (7). We show first that $G \cdot x_n \to G \cdot z$ in X/G. Let $q : X \to X/G$ be the quotient map. Let U be a neighbourhood of $G \cdot z$ in X/G and $V = q^{-1}(U)$. There exists m such that $W_m \subset V$. Since $\liminf_n \nu_{x_n}(q_{x_n}(\phi_{x_n}^{-1}(W_m))) > 0$ there exists n_0 such that $\phi_{x_n}^{-1}(W_m) \neq \emptyset$ for $n \geqslant n_0$. Thus, for $n \geqslant n_0$,

$$G \cdot x_n = q(x_n) \in q(W_m) \subset q(V) = U.$$

Next, suppose that $M_L(\operatorname{Ind}(z,1), (\operatorname{Ind}(x_n,1))) < \infty$. Then, as in the proof of Theorem 5.2, we may localise to an open G-invariant neighbourhood Y of z such that $G \cdot z$ is the unique limit in Y/G of the sequence $(G \cdot x_n)_n$. Eventually $W_m \subset Y$, and so the sequence $(x_n)_n$ converges k-times in Y/G to z by Proposition 3.5 applied to Y. But now $(x_n)_n$ converges k-times in X/G to z as well.

Finally, suppose that $M_L(\operatorname{Ind}(z,1), (\operatorname{Ind}(x_n,1))) = \infty$. By Theorem 5.2, for every open neighbourhood V of z such that $\phi_z^{-1}(V)$ is relatively compact, $\nu_{x_n}(q_{x_n}(\phi_{x_n}^{-1}(V))) \to \infty$ as $n \to \infty$. Let $(K_m)_{m \geqslant 1}$ be an increasing sequence of compact subsets of G such that $G = \bigcup_{m \geqslant 1} \operatorname{Int}(K_m)$ and let $(V_m)_{m \geqslant 1}$ be a decreasing basic sequence of open neighbourhoods of z such that $\phi_z^{-1}(V_1)$ is relatively compact (such neighbourhoods exist by [3, Lemma 2.1]).

For fixed m,

$$\nu_{x_n}\big(q_{x_n}\big(\phi_{x_n}^{-1}(V_m)\big)\big) > (k-1)\nu_z\big(q_z(K_m)\big) + 1 > (k-1)\nu_{x_n}\big(q_{x_n}(K_m)\big)$$

eventually by Lemma 3.4. So there exists a strictly increasing sequence of positive integers n_m such that, for $n \geqslant n_m$,

$$\nu_{x_n}\big(q_{x_n}\big(\phi_{x_n}^{-1}(V_m)\big)\big) > (k-1)\nu_{x_n}\big(q_{x_n}\big(\phi_{x_n}^{-1}(K_m)\big)\big). \tag{6.2}$$

If $n_1 > 1$, then for $1 \leqslant n < n_1$, we set $t_n^{(i)} = e$ for $1 \leqslant i \leqslant k$. For each $n \geqslant n_1$, there is a unique m such that $n_m \leqslant n_{m+1}$. Choose $t_n^{(1)} \in \phi_{x_n}^{-1}(V_m)$. Using (6.2) we have

$$\begin{aligned}
\nu_{x_n}\big(q_{x_n}\big(\phi_{x_n}^{-1}(V_m) \setminus K_m t_n^{(1)} S_{x_n}\big)\big) &\geqslant \nu_{x_n}\big(q_{x_n}\big(\phi_{x_n}^{-1}(V_m)\big) \setminus q_{x_n}\big(K_m t_n^{(1)} S_{x_n}\big)\big) \\
&> (k-1)\nu_{x_n}\big(q_{x_n}(K_m)\big) - \nu_{x_n}\big(q_{x_n}(K_m)\big) \\
&= (k-2)\nu_{x_n}\big(q_{x_n}(K_m)\big).
\end{aligned}$$

So if $k \geqslant 2$ we may choose $t_n^{(2)} \in \phi_{x_n}^{-1}(V_m) \setminus K_m t_n^{(1)} S_{x_n}$. Continuing in this way, we obtain $t_n^{(1)}, \dots, t_n^{(k)} \in \phi_{x_n}^{-1}(V_m)$, such that, for $1 < j \leqslant k$, $t_n^{(j)} \in \phi_{x_n}^{-1}(V_m) \setminus (\bigcup_{i=1}^{j-1} K_m t_n^{(i)} S_{x_n})$. Thus, for $n_m \leqslant n < n_{m+1}$ we have

$$t_n^{(i)} \cdot x_n \in V_m \quad \text{for } 1 \leqslant i \leqslant k \quad \text{and} \quad t_n^{(j)} \notin K_m t_n^{(i)} S_{x_n} \quad \text{for } 1 \leqslant i < j \leqslant k.$$

Therefore, arguing as in Proposition 3.5 (with W_m replaced by V_m) we obtain that $(x_n)_n$ converges k-times in X/G to z.

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